MATHEMATICS BOOK

ACCOUNTING PROFESSION OPTION
for Rwandan Schools

Senior 6 Student Book

Kigali, 2023

FOREWORD

Dear Student,

Rwanda Basic Education Board (REB) is honored to present Senior 6 Mathematics book for the students of Accounting Profession Option which serves as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or in groups.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self-explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the unit title and key unit competence are given and they are followed by the introductory activity before the development of mathematical concepts that are connected to real world problems more especially to production, finance and economics.

The development of each concept has the following points:

- Learning activity which is a well set and simple activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and handling calculations problems not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the development of this book, particularly, REB staff, development partners, Universities Lecturers and secondary school teachers for their technical support. A word of gratitude goes to Secondary Schools Head Teachers, Administration of different Universities (Public and Private Universities) and development partners who availed their staff for various activities.

Any comment or contribution for the improvement of this textbook for the next edition is welcome.

Dr. MBARUSHIMANA Nelson

Director General, REB.

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Joan MURUNGI

Head of CTLR Department

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UNIT 1 APPLICATIONS OF MATRICES AND DETERMINANTS

Key unit competence: Apply matrices and determinants concepts in solving inputs & outputs model and related problems.

Introductory activity

Prices of the three commodities **A,B** and **C** are **X,Y** and **Z** per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 units of B and one unit of C. in the process P, Q and R earn 15000Frw, 1000Frw and 4000Frw respectively. Use matrix inversion to calculate the prices per unit of A, B and C.

1.1. Applications of matrices

1.1.1. Economic applications

Learning activity 1.1.1

Three customers purchased biscuits of different types P, Q and R. The first customer purchased 10 packets of type P, 7 packets of type Q and 3 packets of type R. Second customer purchased 4 packets of type P, 8 packets of type Q and 10 packets of type R. The third customer purchased 4 packets of type P, 7 packets of type Q and 8 packets of type R. If type P costs 4000Frw, type Q costs 5000Frw and type R costs 6000Frw each, then using matrix operation, find the amount of money spent by these customers individually.

Economics is the study of shortage and how it impacts the use of resources, the production of products and services, the growth of production and welfare through time, as well as a wide range of other complicated issues that are of the utmost importance to society in order to satisfy human needs and wants. To understand money, jobs, prices, monopolies, and how the world works on a daily basis, one must study economics. All across the world, matrices' properties, determinants, and inverse matrices, together with their addition, subtraction, and multiplication operations, are employed to address these real-world issues. In economics, matrices are typically also used in the production process.

Example 1

A firm produces three products A, B and C requiring the mix of three materials P, Q and R. The requirement (per unit) of each product for each material is as follows.

$$products(A,B,C)\left\{\overbrace{\begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix}}^{materials(P,Q,R)}\right.$$

Using matrix notation, find

i. The total requirement of each material if the firm produces 100 units of each product

ii. The per unit cost of production of each product if the per unit cost of materials P,Q and R is 5000Frw, 10000Frw and 5000Frw respectively

iii. The total cost of production if the firm produces 200 units of each product

Solution

i. The total requirement of each material if the firm produces 100 units of each product can be calculated using the matrix multiplication given below

$$\begin{bmatrix} 100 & 100 & 100 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 800 & 900 & 800 \end{bmatrix}$$

ii. Let the per unit cost of materials P, Q and R be represented by 3×1 matrix as under $C = \begin{matrix} P \\ Q \\ R \end{matrix}\begin{bmatrix} 5000 \\ 10000 \\ 5000 \end{bmatrix} \rightarrow C = \begin{matrix} P \\ Q \\ R \end{matrix}\begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$

With the help of matrix multiplication, the per unit cost of production of each product would be calculated as

$$AC = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 45 \\ 65 \\ 60 \end{bmatrix}$$

iii. The total cost of production if the firm produces 200 units of each product would be given as

$$\begin{bmatrix} 200 & 200 & 200 \end{bmatrix} \begin{bmatrix} 45 \\ 65 \\ 60 \end{bmatrix} = \begin{bmatrix} 34000 \end{bmatrix}$$

Hence, the total cost of production will be 34000Frw

Example 2

The number of units of a product sold by a retailer for the last 2 weeks are shown in matrix a below, where the columns represent weeks and the rows correspond to the two different shop units that sold them. $A = \begin{pmatrix} 12 & 30 \\ 8 & 15 \end{pmatrix}$ if each item sells for 4000Frw, derive a matrix for total sales revenue for this retailer for these two shop units over this two-week period.

Solution:

Total revenue is calculated by multiplying each element in matrix of sales quantities **A** by the scalar value 4000, the price that each unit is sold at. Thus total revenue can be represented in Frw by the matrix

$$R = (4000)A = \begin{bmatrix} 4000 \times 12 & 4000 \times 30 \\ 4000 \times 8 & 4000 \times 15 \end{bmatrix} = \begin{bmatrix} 48000 & 120000 \\ 32000 & 60000 \end{bmatrix}$$

Example 3

If the price of TV is 300000Frw, the price of a stereo is 250000Frw, the price of a tape deck is 175000Frw. Present price P in a 3×1 matrix then determine the value of stock for the outlet given by $Q = \begin{bmatrix} 20 & 14 & 8 \end{bmatrix}$

Solution

The value of stock is $V = QP$

The price P given by $P = \begin{bmatrix} 300000 \\ 250000 \\ 175000 \end{bmatrix}$

The value of stock is $V = 20(300000) + 14(250000) + 8(175000) = 10,900,000 Frw$

Example 4

A hamburger chain sells 1000 hamburgers, 600 cheeseburgers, and 1200 milk shakers in a week. The price of a hamburger is 450Frw, a cheeseburger 600Frw and a milk shake 500Frw. The cost to the chain of a hamburger is 380Frw, a cheeseburger 420Frw, and a milk shake 320Frw. Find the firm's profit for the week, using

a) total concepts

b) per-unit analysis to prove that matrix multiplication is distributive

Solution

a) The quantity of goods sold (Q), the selling price of the goods (P), and the cost of goods (C) can all be presented in matrix form.

$$Q = \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix} \quad P = \begin{bmatrix} 450 \\ 600 \\ 500 \end{bmatrix} \quad C = \begin{bmatrix} 380 \\ 420 \\ 320 \end{bmatrix}$$

Total revenue(TR)is given by

$$TR = PQ = \begin{bmatrix} 450 \\ 600 \\ 500 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$$

Which is not defined as given. Taking the transpose of P or Q will render the vectors conformable for multiplication. Note that the order of multiplication is important. Thus, taking the transpose of P and premultiplying, we get,

$$TR = P'Q = \begin{bmatrix} 450 & 600 & 500 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$$

$TR = [450(1000) + 600(600) + 500(1200)] = 1,410,000$

Hence, total revenue is 1,410,000Frw

Similarly, total cost (TC) is $TC = C'Q$

$$TC = \begin{bmatrix} 380 & 420 & 320 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix} = [380(1000) + 420(600) + 320(1200)]$$

$TC = 1,016,000 Frw$

Therefore, profit $TR - TC = 1,410,000 Frw - 1,016,000 Frw = 394,000 Frw$

b) Using per-unit analysis, the per-unit profit (U) is

$$U = P - C = \begin{bmatrix} 450 \\ 600 \\ 500 \end{bmatrix} - \begin{bmatrix} 380 \\ 420 \\ 320 \end{bmatrix} = \begin{bmatrix} 70 \\ 180 \\ 180 \end{bmatrix}$$

Total profit is per-unit profit times the number of items sold

$Total\ profit = UQ = \begin{bmatrix} 70 \\ 180 \\ 180 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$ which is undefined. Taking the transpose of U,

the total profit become: $Total\ profit = U'P = \begin{bmatrix} 70 & 180 & 180 \end{bmatrix} \begin{bmatrix} 1000 \\ 600 \\ 1200 \end{bmatrix}$

$= [70(1000) + 180(600) + 180(1200)] = 394,000$

Therefore, the total profit is 394,000Frw

Application activity 1.1.1

A firm produces two goods in a pure competition, and has a following total revenue and total cost functions $TR = 15q_1 + 18q_2$ and $TC = 2q_1^2 + 2q_1q_2 + 3q_2^2$. The two goods are technically related to production, since the marginal cost of one is dependent on the level of output of other. Maximize the profits for the firm using Cramer's rule.

1.1.2. Financial applications

Learning activity 1.1.2

With clear example discuss the importance of matrices in finance and accounting.

The process of raising cash or funds for any kind of expense is called financing. To further handle this process, matrices are more useful in financial applications. Matrix algebra is used in financial calculations due to the large amount of data involved. These include managing financial risk, presenting investment or business expense results or returns, and calculating value-at-risk.

To facilitate a better understanding of accounting procedures based on the principle of double entry, various methods of solving a given problem exist. It is very crucial to emphasize the use of matrix algebra in financial records for accountants, bankers, and accounting students in order to prepare them for emerging trends in the business/financial world.

Application of Matrix Additive to Financial Records

In the application of matrix addition concept, one needs to examine thoroughly the accounting records to obtain the relevant figures expressed in monetary terms to form the required matrices. The principle of double entry in accounting provided the need to record any business transaction twice (i.e. as debit and credit). Accounting records in books are strictly governed by the principle of double entry, making it easier to obtain matrices of financial transactions from one period to the next. As a result, using appropriate matrix concepts, the opening and closing balances can be determined from period to period with little difficulty.

In addition, financial economics and financial econometrics rely on the use of matrix algebra because of the need to manipulate large data inputs. Therefore, matrix used in financial risk management, used to describe the outcomes or payoff of an investment, and used to calculate value at risk.

Example

A finance company has offices located in province, district and cells. Assume that there are 5 provinces, 30 districts and 200 sectors to be considered by the company. The workers at every level are given in the matrix form,

$B=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 2 & 1\end{bmatrix}$, where 1st column stands for number of office supervisor, 2nd column stands for head clerks , and the 3rd column stands for cashiers. The basic monthly salaries (in FRW) are as follows:

Office supervisor: 150,000

Head clerk: 120,000

Cashier: 117,500

Using matrix notation, find

i. The total number of posts of each kind in all the offices taken together

ii. The total basic monthly salary bill of each kind of office and,

iii. The total basic monthly salary bill of all the offices taken together

Answers

The number of offices can be arranged in a row matrix $A=\begin{bmatrix}5 & 30 & 200\end{bmatrix}$

The column matrix D will have the elements that correspond to the basic monthly salary

$$D=\begin{bmatrix}150,000\\ 120,000\\ 117,500\end{bmatrix}$$

i. Total number of posts of each kind in all the offices are the columns of the matrix

$$AB=\begin{bmatrix}5 & 30 & 200\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 2 & 1\end{bmatrix}=\begin{bmatrix}5 & 435 & 235\end{bmatrix}$$

ii. Total basic monthly salary bill of each kind of offices is the elements of matrix

$$BD = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 150,000 \\ 120,000 \\ 117,500 \end{bmatrix} = \begin{bmatrix} 387,500 \\ 237,500 \\ 357,500 \end{bmatrix}$$

iii. The total bill of all the offices is the element of the matrix

$$\begin{bmatrix} 5 & 30 & 200 \end{bmatrix} \begin{bmatrix} 387,500 \\ 237,500 \\ 357,500 \end{bmatrix} = 80,562,500$$

Application activity 1.1.2

Visit the finance office and look at how he/she completes the cash books then perform the following:

a) Create a matrix from debited and credit transactions

b) Find out transaction vector from these matrices

1.2. System of linear equations

1.2.1 Introduction to linear systems

Learning activity 1.2.1

Kalisa bought in Ruhango Market 5 Cocks and 4 Rabbits and he paid 35,000Frw, on the following day, he bought in the same Market 3 Cocks and 6 Rabbits and he paid 30,000Frw.

a) Considering x as the cost for one cock and y the cost of one Rabbit, formulate equations that illustrate the activity of Kalisa;

b) Make a matrix A indicating the number of cocks and rabbits

c) If C is a matrix column made by the money paid by Kalisa, ie

$$C = \begin{pmatrix} 35,000 \\ 30,000 \end{pmatrix}, \text{ write the equation } A\begin{pmatrix} x \\ y \end{pmatrix} = C$$

d) Discuss and explain in your own words how you can determine $\begin{pmatrix} x \\ y \end{pmatrix}$ the cost of 1 Cocks and 1 Rabbit.

1. Linear equation

A line in the $XY-plane$ can be represented by an equation of the form $a_1x+a_2y=b$. This equation is said to be linear in the variables $x\, and\, y$. For example $x+3y=6$, note if $x=0$ then $3y=6$ so $y=2$. Likewise $y=0$ when $x=6$. Thus the line passes through the point $(0,2)$ and $(6,0)$. In general, a linear equation in n variables, $x_1, x_2, x_3..., x_n$ form $a_1x_1+a_2x_2+a_3x_3+...+a_3x_n=b$ where $a_1, a_2, a_3, ..., a_n, b$ are scalars (constants), the variables $x_1, x_2, x_3..., x_n$ can be called unknowns.

A linear equation does not include variable powers or variable products. The variables also do not appear as arguments to exponential or logarithmic functions.

Example

The equation $x+2y+3z=4$ or $x_1+2x_2+3x_3=4$ is a linear equation.

A solution of linear equation is a sequence of n numbers $s_1, s_2, ..., s_n$ such that $x_1=\mathrm{s}_1, x_2=\mathrm{s}_2, ..., x_n=s_n$ makes equation satisfied when replaced in equation. The set of all solution is solution set or general solution.

For example $x+4y=8$, if $x=0$ then $y=2$. Likewise $y=0$ then $x=8$.

2. System of linear equations

A system of linear equations or linear system is a finite set of linear equations in the variables $x_1, x_2, ..., x_n$ or $x, \mathrm{y}, \mathrm{z}, ...$. A solution of system of linear equations is a sequence of n numbers $s_1, s_2, ..., s_n$ such that $x=\mathrm{s}_1, \mathrm{y}=\mathrm{s}_2, \mathrm{z}=s_3, ...$ is a solution of every equation in the system.

Example

$\begin{cases} x+y=2 \\ 2x-y=1 \end{cases}$ This is a system of linear equations in 2 unknowns known as a 2×2 linear system. Graphical, comparison, substitution, or Cramer's methods can be used to solve this system. The same solution of system of linear equations can be found by using matrices.

Note:

Not all systems of linear equations have solution.

For example, $\begin{cases} x+y=4 \\ 2x+2y=6 \end{cases}$

By simplifying the second equation and found that $x+y=3$ is contradictory to the first equation $x+y=4$

A system of linear equations that has no solution is said to be inconsistent. If there is at least one solution it is called consistent. Therefore, every system of linear equations has either no solution, exactly one solution or infinitely many solutions.

An arbitrary system of m equations and n variables ($m \times n$ linear system) can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots + \ldots\ldots\ldots + \ldots\ldots + \ldots\ldots\ldots = \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases},$$

Now, in the above generalized form let us take $m=3$ and $n=3$ then the system become

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

To determine the solutions of this system of linear equation, it is usually write the system in matrix form and augmented matrix for linear system then solve for these variables.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Example 1

Solve the following system of linear equation $\begin{cases} x+y-z=0 \\ x+2y+3z=14 \\ 2x+y+4z=16 \end{cases}$

Solution

Augmented matrix: $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 3 & 14 \\ 2 & 1 & 4 & 6 \end{array}\right) L_2 \sim L_2 - L_1$, $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 14 \\ 2 & 1 & 4 & 6 \end{array}\right) L_3 \sim L_3 - 2L_1$

$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 10 & 30 \end{array}\right)$; from the last matrix we have $10z = 30 \Rightarrow z = 3$

$\Rightarrow y+4z=14 \Rightarrow y+12=14 \Rightarrow y=2$

$\Rightarrow x+y-z=0 \Rightarrow x+2-3=0 \Rightarrow x=1$

$S=\{(1,2,3)\}$

Example 2

Solve the following system of linear equation $\begin{cases}3x+2y+4z=-1\\ 2x-y+2z=-2\\ -x+y+2z=2\end{cases}$

Solution

$$\left(\begin{array}{ccc|c}3 & 2 & 4 & -1\\ 2 & -1 & 2 & -1\\ -1 & 1 & 1 & 2\end{array}\right) L_2 \sim 3L_2-2L_1 \Rightarrow \left(\begin{array}{ccc|c}3 & 2 & 4 & -1\\ 0 & -7 & -2 & -2\\ -1 & 1 & 1 & 2\end{array}\right) L_3 \sim 3L_3-L_1$$

$$\left(\begin{array}{ccc|c}3 & 2 & 4 & -1\\ 0 & -7 & -2 & -4\\ 0 & 5 & 10 & 5\end{array}\right) L_3 \sim 5L_2+7L_3 \Rightarrow \begin{pmatrix}3 & 2 & 4 & -1\\ 0 & -7 & -2 & -4\\ 0 & 0 & 60 & 15\end{pmatrix}$$

$$\Rightarrow 60z=15 \Rightarrow \boldsymbol{z}=\frac{1}{4}$$

$$3x+2y+4z=-1 \Rightarrow 3x+1+1=-1 \Rightarrow 3x=-3 \Rightarrow \boldsymbol{x}=-1$$

$$\Rightarrow -7y-2z=-4 \Rightarrow -7y-\frac{2}{4}=-4 \Rightarrow \boldsymbol{y}=\frac{1}{2}$$

Therefore, $S=\left\{-1;\frac{1}{2};\frac{1}{4}\right\}$

Application activity 1.2.1

Solve the following system of linear equations

a) $\begin{cases}-x+2y=5\\ 2x+3y=4\\ 3x-6y=-15\end{cases}$

b) $\begin{cases}a-3b=-5\\ 3a-b+2c=7\\ 5a-2b+4c=10\end{cases}$

1.2.2 Application of system of linear equations

Learning activity 1.2.2

A factory produces three types of the ideal food processor. Each type X model processor requires 30 minutes of juice processing, 40 minutes of bread processing, and 30 minutes of yoghurt processing, while each type Y model processor requires 20 minutes of juice processing, 50 minutes of bread processing, and 30 minutes of yoghurt processing. Each type Z model processor requires 30 minutes of juice processing, 30 minutes of bread processing, and 20 minutes of yoghurt processing. How many of each type will be produced if 2500 minutes of juice processing, 3500 minutes of bread processing, and 2400 minutes of yoghurt processing are used in one day?

When there is more than one unknown and enough information to set up equations in those unknowns, systems of linear equations are used to solve such applications. In general, we need enough information to set up n equations in n unknowns if there are n unknowns. Solving such system means finding values for the unknown variables which satisfy all the equations at the same time . Even though other methods for solving systems of linear equations exist , one method like the use matrices is the more important choice in economics, finance, and accounting for solving systems of linear equations.

Consider any given situation that represents a system of linear equations, consider the following keys points while solving the problem:

i. Identify unknown quantities in a problem represent them with variables

ii. Write system of equations which models the problem's conditions

iii. Deduce matrices from the system

iv. Solve for unknown variables

Example 1

Mr. John invested a part of his investment in 10% bond A and a part in 15% bond B. His interest income during the first year is 4,000Frw. If he invests 20% more in 10% bond A and 10% more in 15% bond B, his income during the second year increases by 500Frw. Find his initial investment in bonds A and B using matrix method.

Solution

Let the initial investment be x in 10% bond A and y in 15% bond B. Then, according to the given information, we have

$$\begin{cases} 0.10x + 0.15y = 4,000 \\ \left((20\%)(10\%) + 10\%\right)x + \left((10\%)(15\%) + 15\%\right)y = 4,500 \end{cases}$$

$\begin{cases} 0.10x + 0.15y = 4,000 \\ 0.12x + 0.165 = 4,500 \end{cases}$ this is equivalent to $\begin{cases} 2x + 3y = 80,000 \\ 8x + 11y = 3,000,000 \end{cases}$

Express the above equation in matrix form, we obtain

$\begin{pmatrix} 2 & 3 \\ 8 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 80,000 \\ 300,000 \end{pmatrix}$, this can be expressed as $AX = B$ or $X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & 3 \\ 8 & 11 \end{vmatrix} = -2$$

$$X = -\frac{1}{2}\begin{pmatrix} 11 & -3 \\ -8 & 2 \end{pmatrix}\begin{pmatrix} 80,000 \\ 300,000 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -20,000 \\ -40,000 \end{pmatrix} X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10,000 \\ 20,000 \end{pmatrix}$$

Hence $x = 10,000$ and y=20,000Frw and the new investments would be $12,000Frw$, and $22,000Frw$ respectively.

Example 2

A company products three products everyday. Their total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons while the total combined production of the first and the third product is twice that of the second product. Determine the production level of each product using cramer' rule.

Solution

Let the production level of the three products be x, y, z respectively. Therefore, we will have the following equations: $x + y + z = 45$, $z = x + 8$, $x + z = 2y$

The system become $\begin{cases} x + y + z = 45 \\ -x + z = 8 \\ x - 2y + z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\begin{pmatrix} 45 \\ 8 \\ 0 \end{pmatrix}$ which gives,

$\Delta=\begin{vmatrix}1 & 1 & 1\\ -1 & 0 & 1\\ 1 & -2 & 1\end{vmatrix}=6$, since $\Delta\neq 0$ there is a unique solution $\Delta_x=\begin{vmatrix}45 & 1 & 1\\ 8 & 0 & 1\\ 0 & -2 & 1\end{vmatrix}=66$,

$\Delta_y=\begin{vmatrix}1 & 45 & 1\\ -1 & 8 & 1\\ 1 & 0 & 1\end{vmatrix}=90$, $\Delta_z=\begin{vmatrix}1 & 1 & 45\\ -1 & 0 & 8\\ 1 & -2 & 0\end{vmatrix}=114$

Therefore, $x=\frac{66}{6}=11$ $y=\frac{90}{6}=15$ $z=\frac{114}{6}=19$

Hence, the production levels of the products are given by First product has 11 tons, second product has 15 tons and the third product has 19 tons

Application activity 1.2.2

A couple of two person has 60,000,000Frw to invest. They wish to earn an average of 5,000,000 per year on the investment over a 5 year period. Based on the yearly average return on mutual funds for 5 years ending december, 2022, they are considering the following funds: Personal savings at 5%, bank loans at 6%,

a) As their financial advisor, prepare a table showing the various ways the couple can achieve their goal
b) Comment on the various possibilities and the overall plan

1.3. Input-outputs models and Leontief theorem for matrix of order 2.

1.3.1 Input-outputs models ($n=2$)

Learning activity 1.3.1

1. Generate input-output model formula and conditions hold for technology matrix
2. Two commodities A and B are produced such that 0.4 tons of A and 0.7 tons of B are required to produce a tons of A. similarly 0.1 tons of A and 0.7 tons of B are needed to produce a tons of B.

i. Write down a technology matrix
ii. If 6.8 tons of A and 10.2 tons of B are required, find the gross production of both them.

Input Output analysis is a type of economic analysis that focuses on the interdependence of economic sectors. The method is most commonly used to estimate the effects of positive and negative economic shocks and to analyze the unforeseen consequences across an economy. Input-output tables are the foundation of input-output analysis. Such tables contain a series of rows and columns of data that quantify the supply chain for various economic sectors. The industries are listed at the top of each row and column. The information in each column corresponds to the level of inputs used in the production function of that industry. The column for auto manufacturing, for example, displays the resources required to construct automobiles (**i.e.,** requirement of steel, aluminum, plastic, electronic etc,). Input – Output models typically includes separate tables showing the amount of required per unit of investment or production.

The use of Input-Output Models

Input-output models are used to calculate the total economic impact of a change in industry output or demand for one or more commodities. These models employ known information about inter-industry relationships to track all changes in the output of supplier industries required to support an initial increase in an industry's output or an increase in commodity expenditures. The model is commonly shocked during this process.

Let us consider a simple economic model with two industries, A1 and A2, each producing a single type of product. Assume that each industry consumes a portion of its own output and the remainder from the other industry in order to function. As a result, the industries are interdependent. Also, assume that whatever is produced is consumed. That is, each industry's total output must be sufficient to meet its own demand, the demand of the other industry, and the external demand (final demand).The main goal is to determine the output levels of each of the two industries in order to meet a change in final demand, based on knowledge of the two industries' current outputs, under the assumption that the economy's structure does not change.

Let a_{ij} denote the value of output A_i consumed by A_j where $i, j = 1, 2$

Let x_1 and x_2 be the value of the current outputs of A_1 and A_2 respectively.

Let d_1 and d_2 be the value of the final demands for the outputs of A_1 and A_2 respectively.

We construct the two equations based on our generalizations.

$a_{11} + a_{12} + d_1 = x_1$, $a_{21} + a_{22} + d_2 = x_2$ (equation 1)

Let $b_{ij} = \dfrac{a_{ij}}{x_j} \quad i, j = 1, 2$

$$b_{11} = \frac{a_{11}}{x_1}, b_{12} = \frac{a_{12}}{x_2}, b_{21} = \frac{a_{21}}{x_1}, b_{22} = \frac{a_{22}}{x_2}$$

The equation (1) take the form $b_{11}x_1 + b_{12}x_2 + d_1 = x_1 \quad b_{21}x_1 + b_{22}x_2 + d_2 = x_2$

Rearranged as $\begin{matrix}(1-b_{11})x_1 - b_{12}x_2 = d_1 \\ -b_{21}x_1 + (1-b_{22})x_2 = d_2\end{matrix}$

In matrix form of the above equations is

$$\begin{pmatrix} 1-b_{11} & -b_{12} \\ -b_{21} & 1-b_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$(I-B)X = D$, where $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

By solving we get $X = (I-B)^{-1} D$ where matrix B is known as technology matrix

If B is the technology matrix the following conditions hold

i. The main diagonal elements in $I - B$ must be positive

ii. $|I - B|$ must be positive

Example

In an economy there are two industries P_1 and P_2, and the following table gives the supply and demand position in Rwandan Francs.

Production sector	Consumption sector		Final demand	Gross output
	P_1	P_2		
P_1	10000	25000	15000	50000
P_2	20000	30000	10000	60000

Determine the outputs when the final demand changes to 35000 for P_1 and 42000 for P_2

Solution

$a_{11} = 10 \quad a_{12} = 25 \quad x_1 = 50$

$a_{21} = 20 \quad a_{22} = 30 \quad x_1 = 60$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{10000}{50000} = \frac{1}{5}, \quad b_{12} = \frac{a_{12}}{x_2} = \frac{25000}{60000} = \frac{5}{12}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{20000}{50000} = \frac{2}{5}, \quad b_{22} = \frac{a_{22}}{x_2} = \frac{30000}{60000} = \frac{1}{2}$$

The technology matrix given by $B = \begin{pmatrix} 1/5 & 5/12 \\ 2/5 & 1/2 \end{pmatrix}$,

$$I - B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/5 & 5/12 \\ 2/5 & 1/2 \end{pmatrix} = \begin{pmatrix} 4/5 & -5/12 \\ -2/5 & 1/2 \end{pmatrix}$$

The elements of main diagonal are positive. Then, $|I - B| = \begin{vmatrix} 4/5 & -5/12 \\ -2/5 & 1/2 \end{vmatrix} = \frac{7}{30}$

. Main diagonal elements of $I - B$ are positive and $|I - B|$ is positive. Therefore, the problem has solution.

$Adj(\mathrm{I-B}) = \begin{pmatrix} 1/2 & 5/12 \\ 2/5 & 4/5 \end{pmatrix}$ then, $(I - B)^{-1} = \frac{1}{|I - B|} adj(I - B)$

$$(I - B)^{-1} = \frac{30}{7} \begin{pmatrix} 1/2 & 5/12 \\ 2/5 & 4/5 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 15 & 25/2 \\ 12 & 24 \end{pmatrix}$$

$X = (I - B)^{-1} D$ where $D = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$, $X = \frac{1}{7} \begin{pmatrix} 15 & 25/2 \\ 12 & 24 \end{pmatrix} \begin{pmatrix} 35000 \\ 42000 \end{pmatrix} = \begin{pmatrix} 150000 \\ 204000 \end{pmatrix}$

The output of industry P_1 should be 150000Frw and P_2 should be 204000Frw

Application activity 1.3.1

An economy produces only coal and steel. These to commodities serve as intermediate inputs in each other's production. 0.4 tons of steel and 0.7 tons of coal are needed to produce a ton of steel. Similarly, 0.1 tons of steel and 0.6 tons of coal are required to produce a ton of coal. No capital inputs are needed. Do you think that the system is viable? The 2 and 5 labor days are required to produce a ton of coal and steel respectively. If economy needs 100 tons of coal and 50 tons of steel, calculate the gross output of the two commodities and the total labor days required.

1.3.2 Leontief theorem for matrix of order two

Learning activity 1.3.2

1. Differentiate open Leontief Model from closed Leontief Model
2. The following are two firms that are producing units in tons for agriculture and manufacturing.

Production/Demand	Agriculture	Manufacturing
Agriculture	0.4	0.02
Manufacturing	0.12	0.19

Assume that the external demand of agriculture is 80 and the one of manufacturing is 200. Determine the units to be produced by agriculture and manufacturing firms in order to satisfy their own external demands.

The Leontief model is an economic model for an entire country or region. In this model, **n** industries produce a number (n) of different products such that the input equals the output, or consumption equals production. It distinguishes between closed and open model.

The open Leontief Model is a simplified economic model of a society in which consumption equals production, or input equals output. Internal Consumption (or internal demand) is defined as the amount of production consumed within industries, whereas External Demand is the amount used outside of industries. In addition, some production consumed internally by industries, rest consumed by external bodies. This model allows us to calculate how much production is required in each industry to meet total demand. To determine the production level, first create a consumption matrix C.

In this matrix, each entry c_{ij} represents the number of units of output from industry i required (or consumed) to produce one unit of output of industry j. Then, construct the external demand vector D which gives the external demand for each industry. Let C be the consumption matrix, D be the demand vector, and X be the production vector. Then, CX is the internal consumption.

Starting with *Production = Total demand(internal and External)*, solve for X.

$X = CX + D$, in the open Leontief model C *and* $D \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are given and the problem is to determine X from this matrix equation $X - CX = D$, $(I - C)X = D$, $X = (I-C)^{-1}D$, if $I - C$ is invertible then, $(I-C)^{-1}$ is then called the Leontief inverse.

Therefore, an industry is profitable if the corresponding column in C has sum less than one.

Example

An economy has the two industries R and S. The current consumption is given by the table

	Consumption		
	R	S	External
Industry R production	50	50	20
Industry S production	60	40	100

Assume the new external demand is 100 units of R and 100 units of S. Determine the new production levels.

Solution

The total production is 120 units for R and 200 units for S.

We obtain $X = \begin{pmatrix} 120 \\ 200 \end{pmatrix}$, $B = \begin{pmatrix} 20 \\ 100 \end{pmatrix}$, $A = \begin{pmatrix} 50/120 & 50/200 \\ 60/120 & 40/120 \end{pmatrix}$, and $B' = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$.

The solution is given by $X' = (I_2 - A)^{-1} B' = \frac{1}{41}\begin{pmatrix} 96 & 30 \\ 60 & 70 \end{pmatrix}\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 307.3 \\ 317 \end{pmatrix}$

The new production levels are 307.3 and 317 for R and S, respectively.

The closed Leontief Model

The closed Leontief model can be described by the matrix equation $X = AX$. It means that there is no external demand. The matrix, $I_n - A$ is usually not invertible, otherwise the only solution would be $X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore, there is only internal consumption.

Example

Suppose that an economy has two sectors, Mining and electricity. For each unit of output, Mining requires 0.4 unites of its own production and 0.2 units of electricity. Moreover, for each unit output, electricity requires 0.2 units of Mining and 0.6 units of its own production.

i. Determine the consumption matrix C for this economy

ii. Find the inverse $(I - C)$

iii. Using the Leontief model, determine the production levels from each sector that are necessary to satisfy a final demand of 20 units from mining and 10 units from electricity.

Solution

i. $C = \begin{pmatrix} 0.4 & 0.2 \\ 0.2 & 0.6 \end{pmatrix}$

ii. $I - C = \begin{pmatrix} 0.6 & -0.2 \\ -0.2 & 0.4 \end{pmatrix} = \begin{pmatrix} \frac{6}{10} & -\frac{2}{10} \\ -\frac{2}{10} & \frac{4}{10} \end{pmatrix}$

$$(I - C)^{-1} = \frac{1}{\left(\frac{6}{10}\right)\left(\frac{4}{10}\right) - \left(\frac{-2}{10}\right)\left(\frac{-2}{10}\right)}\begin{pmatrix} \frac{4}{10} & \frac{2}{10} \\ \frac{2}{10} & \frac{6}{10} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

Therefore,

$$(I-C)^{-1}=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$$

iii. Let $D=\begin{pmatrix}20\\ 10\end{pmatrix}$, the Leontief Model states that $CX+D=X$ or equivalent to

$X=(I-C)^{-1}D=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}20\\ 10\end{pmatrix}=\begin{pmatrix}50\\ 50\end{pmatrix}$ Thus, the productiion level needed to meet the demand D is $\begin{pmatrix}50\\ 50\end{pmatrix}$, or 50 units from Mining and 50 units from electricity.

Application activity 1.3.2

An economy has two sectors namely electricity and services. For each unit of output, electricity requires 0.5 units from its own sector and 0.4 units from services. Services require 0.5 units from electricity and 0.2 from its own sector to produce one unit of services.

i. Determine the consumption matrix C

ii. State the Leontief input-output equation relating C to the production X and final demand D

iii. Use an inverse matrix to determine the production necessary to satisfy a final demand of 1000 units of electricity and 2000 units of services.

1.4 End unit assessment

1. Consider the following system:

$$\begin{cases}x+2y-3z=0\\ 3x+3y-z=5\\ x-2y+2z=1\end{cases}$$

a) Rewrite this system in matrix form $AX=B$

b) By using matrices, solve the above system of linear equations

c) Compare the values obtained in (b) to any other methods previously seen.

2. An automobile company uses three types of steel s_1, s_2 and s_3 for producing three types of c_1, c_2 and c_3. The steel requirement for each type of car are:

Cars				
		c1	c2	c3
Steel(in tons)	s1	2	3	4
	s2	1	1	2
	s3	3	2	1

Determine the number of cars of each type which can be produced using 29, 13 and 16 tons of steel of the three types respectively.

3. A company produces three products every day. Their total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons while the total combined production of the first and the third product is twice that of the second product. Determine the production level of each product using Cramer's rule.

4. An economy has two sectors, mining and electricity. For each unit of output, mining requires 40% units of its own production and 20% units of electricity. Moreover, for each unit output, electricity requires 20% units of mining and 0.6 units of its own production.

a) Determine the consumption matrix C for this economy

b) Find the inverse of I-C

c) Using the Leontief model, determine the production levels from each sector that are necessary to satisfy a final demand of 20 units from mining and 10 units from electricity.

UNIT 2 LINEAR INEQUALITIES AND THEIR APPLICATION IN LINEAR PROGRAMMING PROBLEMS

Key unit competencies: Solve linear programming problems

Introductory activity

1. Formulate 3 examples of linear inequalities, then perform the following:
   (a) Form a system of linear inequalities. (b) Represent them on Cartesian plane. (c) Show their meet region. (d) Establish the linkage between them.
2. The two production companies produce two different drinking products that, after being prepared, are graded into three classes: high, medium, and low-quality. The companies agreed to provide 1200 dozens of high-quality, 800 dozens of medium-quality, and 2400 dozens of low-quality drinks per day. Both companies had different operating procedures and took weekends off. Their specifics are provided below.

Company	Cost per hour C"000"	Production quality		
		High	Medium	Low
A	180	600	300	400
B	160	100	100	600

Apply the linear inequality concepts to present the above information and determine the hour per day each company be operated to fulfill the signed contract.

2.1 Recall of linear inequalities

Learning activity 2.1

1. Find the value(s) of x such that the following statements are true

   a. $x < 5$ b. $x > 0$ c. $-4 < x < 12$

2.

   a) With clear examples differentiate linear equations from linear inequalities

   b) Solve the equations and inequalities found in 2 (a).

   c) How these linear inequalities are applied when solving real word problems?

3. Sam and Alex play in the same team at their school. Last Saturday their team played with another team from other school in the same district, Alex scored 3 more goals than Sam. But together they scored less than 9 goals. What is the possible number of goals Alex scored?

Inequality

The statement $x+3=10$ is true only when $x=7$. If x is replaced by 5, we have a statement 5+3=10 which is **false**.

Suppose that we have the inequality $x+3<10$, in this case we have an inequality with one unknown. Here the real value of x satisfies this inequality is not unique. Now, the solution set of $x+3<10$ is an open interval containing all real numbers less than 7 whereby 7 is excluded. Then, we solve this inequality as follow:

$$x+3<10$$
$$\Leftrightarrow x<10-3$$
$$\Rightarrow x<7$$

and then,

$$S=]-\infty,7[$$

Note that:

- When the same real number is added or subtracted from each side of inequality the direction of inequality is not **changed**.

- The direction of the inequality is **not changed** if both sides are multiplied or divided by the same **positive real number** and is **reversed** if both sides are multiplied or divided by the **same negative real** number.

Intervals

A subset of real line is called an interval if it contains at least two numbers and also, it contains all real numbers between any two of its elements. For example, the set of real numbers x such that $x > 6$ is an interval, but the set of real numbers y such that $y \neq 0$ is not an interval.

If a and b are real numbers and $a < b$, we often refer to:

i. The open interval from a to b, denoted by (a,b) or $]a,b[$, consisting of all real numbers x satisfying $a < x < b$

ii. The closed interval from a to b, denoted by $[a,b]$, consisting of all real numbers $x x$ satisfying $a \leq x \leq b$

iii. The half-open interval $[a,b[$, consisting of all real numbers x satisfying the inequalities $a \leq x < b$

iv. Half-open interval $]a,b]$, consisting of all real numbers x satisfying the inequalities $a < x \leq b$

Examples

1. Solve the following inequalities and express the solution in terms of intervals.

a) $2x-1 > x+3$; **b)** $-\frac{x}{3} \geq 2x-1$; **c)** $2(x+5) \geq 2x-8$; **d)** $2x+5 \leq 2x+4$

Solution

a)
$$\begin{aligned} &2x-1 > x+3 \\ &\Rightarrow 2x > x+3+1 \\ &\Rightarrow 2x-x > 4 \\ &\Rightarrow x > 4 \end{aligned}$$

The solution set is the interval $]4,\infty[$

b) $-\frac{x}{3} \geq 2x-1; -x \geq 3(2x-1);\ x \leq -6x+3; 7x \leq 3; x \leq \frac{3}{7};$

The solution set is the interval $\left]-\infty, \frac{3}{7}\right]$

c) $2(x+5) \geq 2x-8; -2x+10 \geq 2x-8; -4x \leq -18; x \geq \frac{9}{2};$

The solution set is the interval $S = \left[\frac{9}{2}, +\infty\right[$

d) $2x+5 \leq 2x+4; 2x-2x \leq 4-5,\ 0x \leq -1.$

Since any real number times zero is zero and zero is not less or equal to -1, then the solution set is the empty set. $S = \phi$

Products / Quotients of inequalities

Suppose that we need to solve the inequality of the form $(ax+b)(cx+d)<0$. For this inequality we need the set of all real numbers that make the left hand side to be negative. Suppose also that we need to solve the inequality of the form $(ax+b)(cx+d)>0$. For this inequality we need the set of all real numbers that make the left hand side to be positive. We follow the following steps:

a) First we solve for $(ax+b)(cx+d)=0$

b) Construct the table called **sign table**, find the sign of each factor and then the sign of the product or quotient if we are given a quotient.

For the quotient the value that makes the denominator to be zero is always excluded in the solution. For that value we use the symbol || in the row of quotient sign.

c) Write the interval considering the given inequality sign.

Examples

In set of real numbers, solve the following inequalities:

a) $(3x+7)(x-2)<0$

b) $\frac{x+4}{2x-1} \geq 0$

Solution

a) $(3x+7)(x-2)<0$; Start by solving the equation $(3x+7)(x-2)=0$, we change inequality into equality signs, and find the solution in which the given product shall be equal to zero.

$$(3x+7)(x-2)=0; \Leftrightarrow (3x+7)=0 \Rightarrow 3x=-7 \Rightarrow x=\frac{-7}{3}; OR, (x-2)=0 \Rightarrow x=2$$

Then , we construct the table of sign of $(3x+7)(x-2)<0$.

x	$-\infty$ $\frac{-7}{3}$ 2 $+\infty$
$(3x+7)$	- 0 + + + + + + +
$(x-2)$	- - - - - 0 + + + +
$(3x+7)(x-2)$	+ + + 0 - - - 0 + + + +

Since the inequality is $(3x+7)(x-2)<0$ we will take the interval where the product is negative. Thus, $S=\left]\frac{-7}{3},2\right[$

b) $\frac{x+4}{2x-1}\geq 0$, this can be solved as $x+4=0 \Rightarrow x=-4, and\, 2x-1=0 \Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$

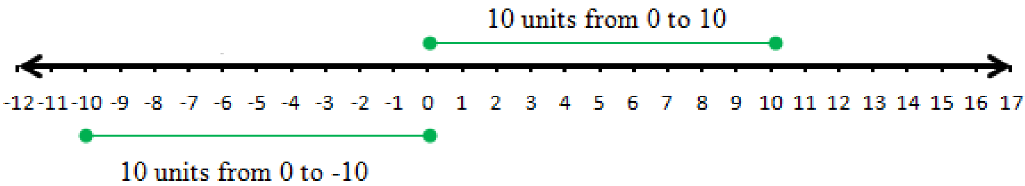
x	$-\infty$ -4 $\frac{1}{2}$ $+\infty$
$x+4$	- 0 + +
$2x-1$	- - 0 +
$\frac{x+4}{2x-1}$	+ 0 - || +

$$S = \left]-\infty, -4\right] \cup \left]\frac{1}{2}, +\infty\right[$$

Inequality and absolute value

The inequality $|x-a| < k$ says that the distance from x to a is less than k so x must lie between $a-k$ and $a+k$ or equivalently a must lie between $x-k$ and $x+k$ if k is a positive number.

Recall that absolute value of a number is the number of units from zero to a number line. That is, $|x| = k$ means k units from zero (k is a positive real number or zero).



For all real number x and $k \geq 0$

a) $|x| = k \Leftrightarrow -k < x < k$

b) $|x-a| < k \Leftrightarrow -k < x-a < +k \Leftrightarrow a-k < x < a+k$

c) $|x| > k \Leftrightarrow x > k, or\ x < -k$

d) $|x-a| > k \Leftrightarrow x > a+k \quad or \quad x < a-k$

Example1

1. Solve $|3x-2| \leq 1$

Solution:

$-1 \leq 3x-2 \leq 1 \Leftrightarrow -1+2 \leq 3x \leq 1+2 \Leftrightarrow 1 \leq 3x \leq 3 \Leftrightarrow \frac{1}{3} \leq x \leq 1$, this is the same as to solve this pair of inequalities $-1 \leq 3x-2$ and $3x-2 \leq 1$, this give us $x \geq \frac{1}{3}$ and $x \leq 1$. Thus the solution lies in the interval $\left[\frac{1}{3}, 1\right]$

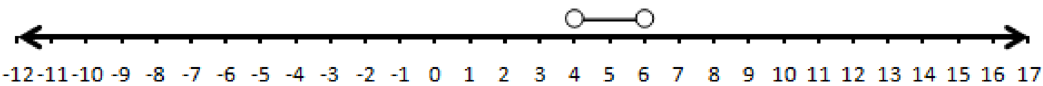
2. Find the solution set of the inequality $|3x-15| < 3$, then:

$$|3x-15|<3 \Leftrightarrow -3<3x-15<3 \Rightarrow -3<3x-15 \Rightarrow -3+15<3x \Rightarrow 3x>12 \Rightarrow x>4$$

$$or \quad 3x-15<3 \Rightarrow 3x<3+15 \Rightarrow 3x<18 \Rightarrow x<6,$$

So, $4<x<6$, *the solution is* $S=\{x \in IR : 4<x<6\}$,

Number line:



Problems involves linear inequalities

Inequalities can be used to model a number of real life situations. When converting such word problems into inequalities, begin by identifying how the quantities are relating to each other, and then pick the inequality symbol that is appropriate for that situation. When solving these problems, the solution will be a range of possibilities. Absolute value inequalities can be used to model situations where margin of error is a concern.

Examples

1. The width of a rectangle is 20 meters. What must the length be if the perimeter is at least 180 meters?

Solution:

Let x be length of rectangle. $Perimeter\ (P) = 2 \times length(L) + 2 \times width(W)$.
$2x+2(20) \geq 180 \Rightarrow 2x \geq 180-40 \Rightarrow 2x \geq 140 \Rightarrow x \geq 70$

$Solution;\ S=\{x \in \mathbb{R} : x \geq 70\}$; $S=[70,+\infty[$

We can conclude that the length must be at least 70meters.

2. John has 1, 260, 000 Rwandan Francs in an account with his bank. If he deposits 30, 000 Rwanda Francs each week into the account, how many weeks will he need to have more than1, 820, 000 Rwandan Francs on his account?

Solution:

Let *x* be the number of weeks, we have :

Then, the total amount of deposits to be made + the current balance > total amount wanted.

That is $30,000x + 1,260,000 > 1,820,000$

$30,000x > 560,000$; $\Rightarrow x > \dfrac{560,000}{30,000} \Rightarrow x > \dfrac{56}{3} \Rightarrow x = 18.6 \simeq 19$

Thus, John need at least 19 weeks to have more than 1, 820, 000 Rwandan Francs on his account.

Application activity 2.1

1) Joyce enters a race in which she must cycle and run. She cycles 25 kilometers and then runs 20 kilometers. Her average running speed is half that of his cycling speed. What can we conclude about Joyce's average speeds after he finishes the race in less than 2½ hours?

2) Visit the library to find out the use of linear inequalities in accounting

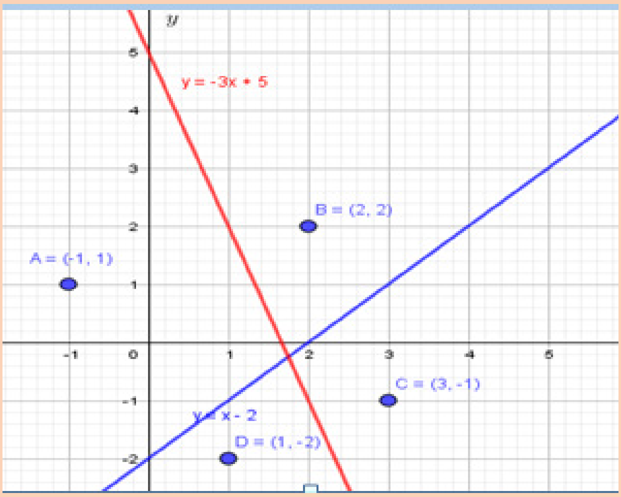
2.2 Basic concepts of linear programming problem (LPP)

2.2.1 Definition and keys concepts

Learning activity 2.2.1

The following graph illustrate two lines and their equations, for each point A, B, C, and D, replace its coordinate in the two inequalities to verify which one satisfies the following system:

$$\begin{cases} y < x - 2 \\ y > -3x + 5 \end{cases}$$



1. What are your observations on the graph as a future accountant?
2. Show the solution of these linear inequalities.
3. What is the common solution?
4. Where do we need this solution in Economics?

- A system of inequalities consits of a set of two or more inequalities with the same variables. The inequalities define the conditions that are to be considered simultaneously.
- Each inequality in the set contains infinetely many ordered pair solutions defined by a region in rectangular coordinate plane. When considering two of these inequalities together, the intersection of these sets will define the set of simultaneous ordered pair solutions.

- A system of linear inequalities is similar to a system of linear equations, except that it is made up of inequalities rather than equations. To model scenarios with multiple constraints, systems of linear inequalities are used.

In finding solution, first, graphs the "equals" line, then shade in the correct area. The following steps will be considered to find the solution of simultaneous linear inequalities with two unknowns graphically, but for more than two is impossible:

1. Rearrange the equation so "y" is on the left and everything else on the right.

2. Plot the Y line (make it a solid line for $y \geq or\ y \leq$ and a dashed line for $y < or\ y >$

3. Shade above the line for a greater than $y > or\ y \geq$ or below the line for a less than $y < or\ y \leq$ The intersection will define the set of simultaneous ordered pair solutions.

Linear inequalities with more than two unknowns are solved to find a range of values of three or more unknowns which make the inequalities true at the same time. The solution is not represented graphically but we can apply one of the following methods we expended in senior five Accounting, include Gaussian elimination, comparison, substitution, or matrix inversion methods, and you are advised to review for recall on yourself.

Example of system:

Solve the system of inequalities by graphing:

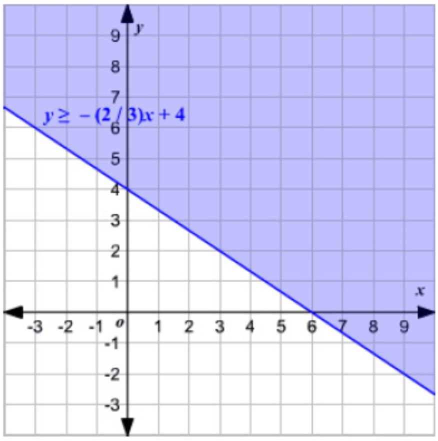
$$\begin{cases} 2x + 3y \geq 12 \\ 8x - 4y > 1 \\ x < 4 \end{cases}$$

Solution using graphical representation:

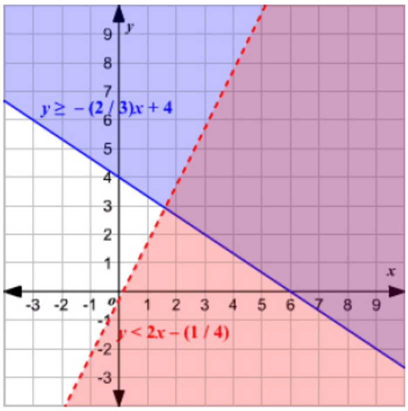
Graph each inequality from the system, we have $2x + 3y \geq 12 \Rightarrow y \geq -\frac{2}{3}x + 4$,

the related equation to this inequality is $y \geq -\frac{2}{3}x + 4$, since the inequality is $\geq$, not a strict one, the border line is solid.

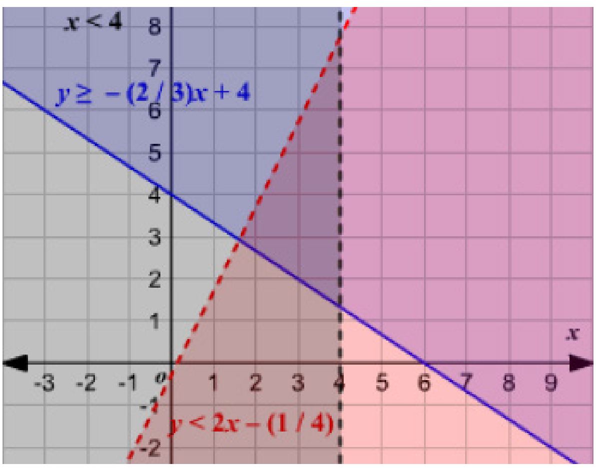
Graph the line $\Rightarrow y = -\frac{2}{3}x + 4$



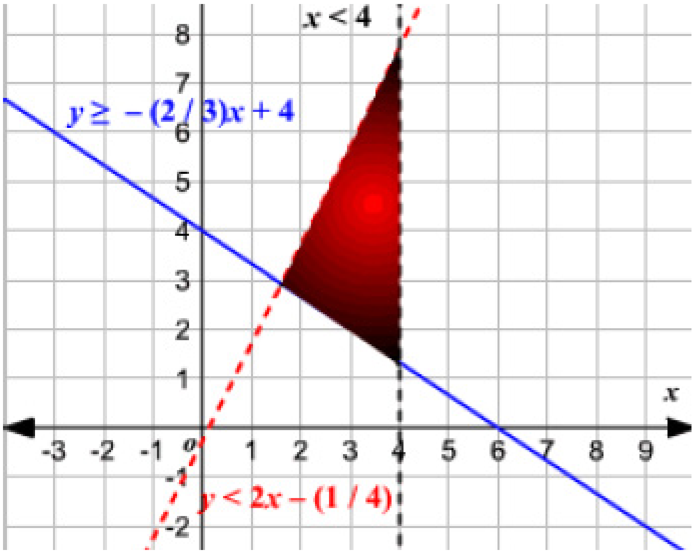
Similarly, we draw a dashed line of related equation of the second inequality $y < 2x - \frac{1}{4}$ which has a strict inequality.



Draw the dashed vertical line $x = 4$ which is the result related to the equation of the 3rd inequality.



The solution of the whole system of inequalities is provided by the intersection region of the solutions of all the three inequalities as it is shown in the figure below.



Linear programming is a subset of Operations Research (OR), an interdisciplinary branch of formal science that employs methods such as mathematical modeling and algorithms to find optimal or near-optimal solutions to complex problems. It has the goal of optimizing the solution (minimize cost or maximize profit).

Therefore, Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equalities and/or inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner.

Application activity 2.2.1

1) Solve graphically the following simultaneous inequalities

a) $\begin{cases} y - 2x \le 1 \\ x + y \le 10 \\ x \ge 0 \end{cases}$ b) $\begin{cases} -3x + 2y > 6 \\ 6x - 4y > 8 \end{cases}$

2) Let us consider any transportation company in Rwanda which has two types of buses, named A and B employs to transport passengers throughout Rwanda. Bus A is capable of transporting 50 passengers and 40 tons of luggage, while Bus B can only fit 70 passengers and 25 tons of luggage. This transportation provider has a contract to transport at least 970 passengers and 370 tons of luggage each trip. If operating bus, a costs FRW1,100,000 per trip and operating bus B costs FRW 1,300,000 per trip: **(i)** Create any linear inequalities related to this problem; **(ii)** What are your observations in terms of graphical method? **(iii)** Is there any purpose? If so, which one? **(iv)** Establish the variables under consideration.

2.2.2 Mathematical models formulation and optimal solution

Learning activity 2.2.2

Refer to the application activity 2.2.1 where the transportation company has signed a contract to transport at least 970 passengers and 370 tons of luggage each trip and transport passengers in Rwanda with Bus A which is capable of transporting 50 passengers and 40 tons of luggage, while Bus B can only fit 70 passengers and 25 tons of luggage. If operating bus A costs FRW1, 100,000 per trip and operating bus B costs FRW 1, 300,000 per trip. Then perform the following:

i. Without computation, which bus option will result in the lowest overall cost per travel? Why?

ii. How can you reduce expenses?

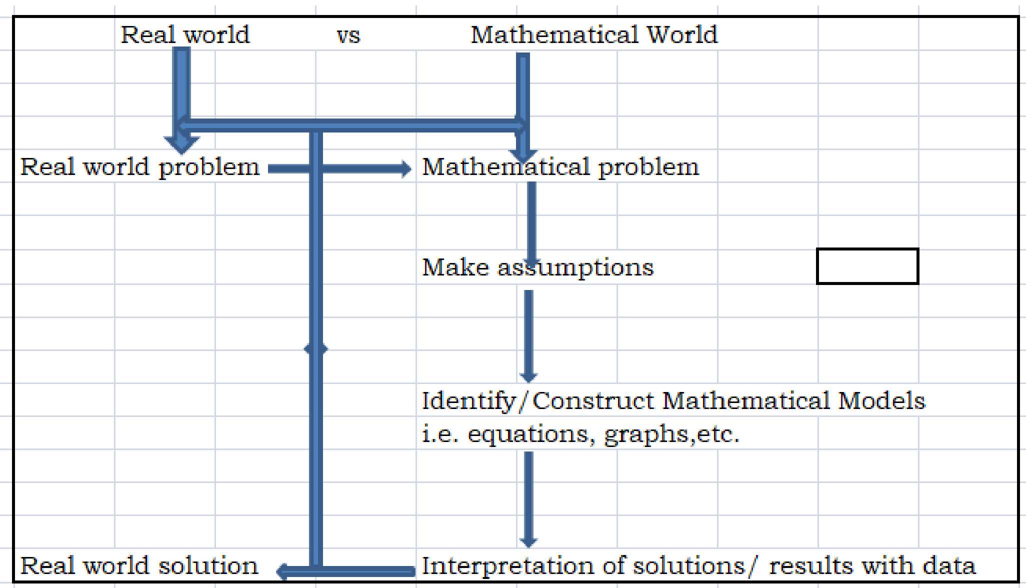
A model in mathematics is simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation are considered.

The objective of a model is to provide a means for analyzing the behaviour of the system for the purpose of improving its performance. There are several models in each area of business, or industrial activity. For instance, an ***account model*** is a typical budget in which business accounts are referred to with the intention of providing measurements such as rate of expenses, quantity sold, etc.; A mathematical equation may be considered to be a ***mathematical model*** in which a relationship between constants and variables is represented.

A model which has the possibility of measuring observations is called a ***quantitative model***; a product, a device or any tangible thing used for experimentation may represent a ***physical model***.

In this course Mathematical Model is used to model an object or situation from the real world using mathematical ideas. It helps us understand the real world and is used to improve many aspects of our lives. Mathematical models are an essential part of the working world, from safety to planning and construction. More broadly, a model is a representation of an object or idea that is used to gain a better understanding of the real thing.

The procedures of translating any real world to mathematical problems are in the table below.



Objective Function

The objective function is a mathematical equation that describes the production output target that corresponds to the maximization of profits with respect to production. It attempts to maximize profits or minimize losses based on a set of constraints and the relationship between one or more decision variables. It is given by $Z = ax + by$

where **a**, **b** are constraints, variables x and y are subject to constraints described by linear inequalities which have to be maximized or minimized. The variables x and y are called the decision variables. An objective function is governed by a few constraints, some of which are $x \geq 0$, $y \geq 0$.

Constraints

The objective function is subject to certain constraints, expressed by linear inequalities. These constraints are actually the limitations on the primary decision variables.

$$\begin{cases} a_1x + b_1y \leq c_1 \\ a_2x + b_2y \leq c_2 \\ a_3x + b_3y \leq c_3 \\ \ldots\ldots \\ a_nx + b_ny \leq c_n \end{cases}$$

Each inequality constraint system determines a half-plane.

Feasible Solution

The feasibility region of the linear programming problem shows set of all feasible solutions. A feasible solution of the linear programming problem satisfies every constraint of the problem.

Corner points

Are the set of all points near feasible solution, and in these points we have to choice the point which help us to maximize or minimize the objective function as an optimal point.

Optimal Solution

It is also included in the feasible region; however, it represents the maximum objective value of the function for the problem which requires maximization and smallest objective value of the function that requires minimization. There are many methods to solve linear programming problems. These methods include a graphing method, Lagrange multipliers method, simplex method, northwest corner method, and least cost method, etc. At this level, students will see only how to solve problems using the graphical method only. Other methods will be studied at the university level.

Example 1:

A furniture dealer has to buy chairs and tables and he has total available money of $50,000 for investment. The cost of a table is $2500, and the cost of a chair is $500. He has storage space for only 60 pieces, and he can make a profit of $300 on a table and $100 on a chair. Express this as an objective function and also find the constraints.

Solution:

Let us consider the number of tables as x and the number of chairs as y. The cost of a table is $2500, and the cost of a chair is $500, and the total cost cannot exceed more than $50,000.

Constraint - I: $2500x + 500y < 50000$ OR $5x + y < 100$.

The dealer does not have storage space for more than 60 pieces. This can be represented as a second constraint.

Constraint - II: $x + y < 60$

There is a profit of $300 on the table and $100 on the chair. The aim is to optimize the profits and this can be represented as the objective function.

Objective Function: $Z = 300x + 100y$.

Therefore, the constraints are $5x + y < 100$, $x + y < 60$, and the objective function is $Z = 800x + 1200y$.

Example 2:

A furniture company produces office chairs and tables. The company projects the demand of at least 100 chairs and 50 tables daily. The company can produce no more than 120 chairs and 70 tables daily. The company must ship at most 150 units of chairs and tables daily to fulfill the shipping contract. Each sold table results in the profit of 50 and each chair produces15 profit

a) How many units of chairs and tables should be made daily to maximize the profit?
b) Compute the maximum profit the company can earn in a day?

Solution

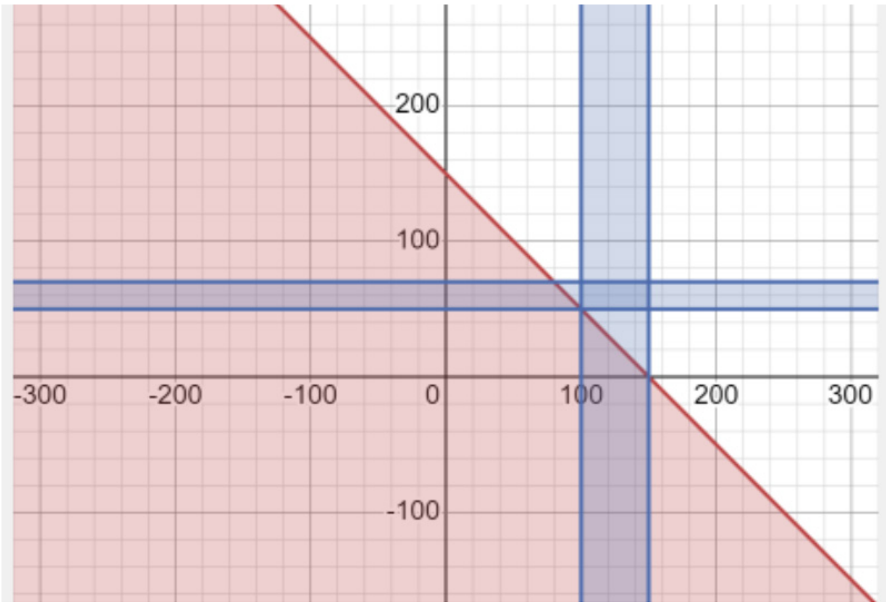
a) Let x stands for the number of chairs sold daily, y stands for the number of tables sold daily.

The objective function is given by $P = 15x + 50y$, The constraints can be $100 \le x \le 150$; again, the problem says that "there is a projected demand for at least 50 tables daily and the company cannot produce more than 70 tables. Which is $50 \le y \le 70$. To satisfy the shipping contract, the company must ship at most 150 units of both chairs and tables daily, and the related constraint is: $x + y \le 150$. Then, mathematical model can be:

$$Maximize\ P = 15x + 50y$$

$$subject\ to\ the\ constra\text{int}\, s \begin{cases} x + y \le 150 \\ 50 \le y \le 70 \\ 100 \le x \le 150 \end{cases}$$

We graph the constraints inequalities on a Cartesian plane, and we obtain the following:



At point (100, 50) is there only one feasible solution which is also the optimal solution of the problem because all the three lines of the graph are intersecting each other. Therefore, the point (100, 50) satisfies all the constraints of the problem. Hence, a furniture company must produces 100 chairs and 50 tables daily to optimize the profits.

b) The maximum profit can be computed by replacing the optimal point in the objective function. For x = 100 and y = 50 in the following equation,

$$P_{(x=100, y=50)} = 15(100) + 50(50) = 4000$$

Application activity 2.2.2

A manufacturing company makes two kinds of instruments. The first instrument requires 9 hours of fabrication time and one hour of labor time for finishing. The second model requires 12 hours for fabricating and 2 hours of labor time for finishing. The total time available for fabricating and finishing is 180 hours and 30 hours respectively. The company makes a profit of 800,000 Frw on the first instrument and 1200, 000Frw on the second instrument. Express this linear programming problem as an objective function and also find the constraint involved.

2.3 Methods of solving linear programming problems (LPP)

2.3.1. Solving LPP using graphical methods

Learning activity 2.3.1

Learning activity 2.3.1

Consider the following problem

c) Which steps can you list to find solution? Graphically solve the given system of linear equation.

d) Do you think Profit can be shown from this graph?

$$\begin{cases} x+y\geq 1 \\ x+2y\leq 6 \\ 2x+y\leq 6 \\ x\geq 0, y\geq 0 \end{cases}$$

To solve LPP we use graphical method under different steps. These are summarized below:

Step 1. Identify the problem-the decision variables, the objective and the restrictions.

Step 2. Set up the mathematical formulation of the problem

Step 3. Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions bounded (represented) by the constraints of the problem and is restricted to the first quadrant only.

Step 4. The feasible region obtained in step 3 may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

Step 5. Find out the value of the objective function at each corner (solution) point determined in step 4

Step 6. Select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the optimum feasible solution.

Example

A factory produces two types of devices including regular and premium. Each device necessitates two operations, assembly and finishing, and each operation has a maximum time limit of 12 hours. A standard device requires 1 hour of assembly and 2 hours of finishing, whereas a premium device requires 2 hours of assembly and 1 hour of finishing. Due to other constraints, the company can only produce 7 devices per day. How many of each should be manufactured to maximize profit if a profit of $20 is realized for each regular device and a profit of $30 for each premium device?

Answer:

Let x be the number of regular device manufactured each day, and y be the number of premium device manufactured each day. The objective function is $P = 20x + 30y$,

The constraint is $x + y \leq 7$ Since the regular device requires one hour of assembly and the premium device requires two hours of assembly, and there are at most 12 hours available for this operation, we get $x + 2y \leq 12$

Similarly, the regular device requires two hours of finishing and the premium device one hour. Again, there are at most 12 hours available for finishing. This gives us the constraint: $2x + y \leq 12$

The fact that x, y cannot be negative, these two constraints are denoted as: $x \geq 0, y \geq 0$.

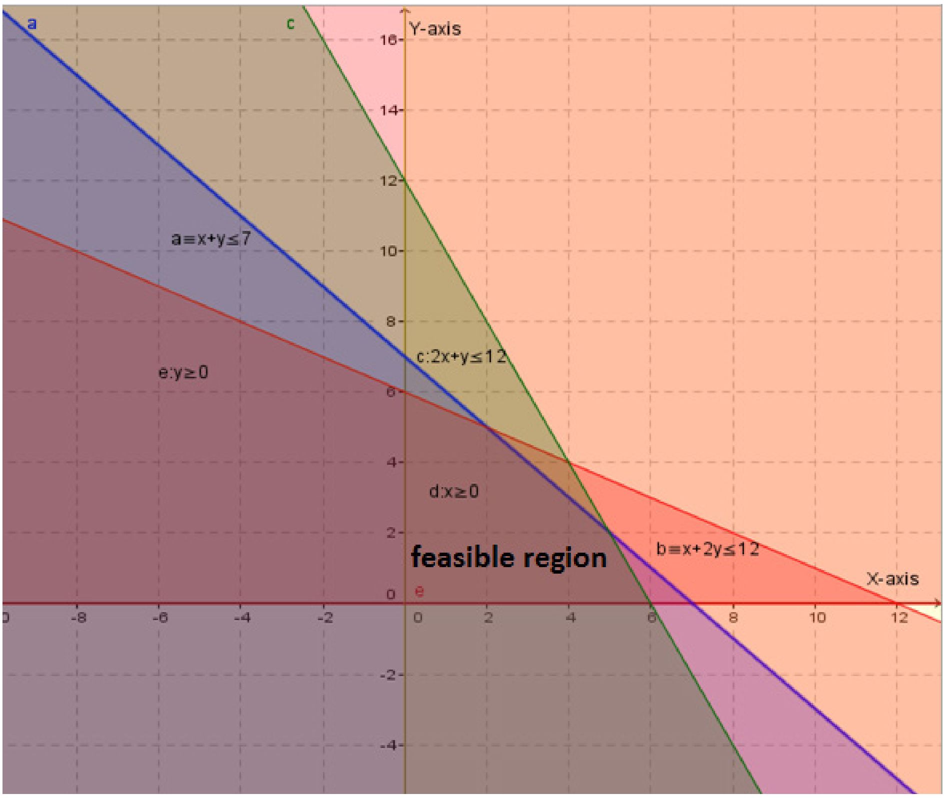
Now, the mathematical problem model can be formulated as follows:

$$Maximize\ P = 20x + 30y$$

$$Subject\ to \begin{cases} x + y \leq 7 \\ x + 2y \leq 12 \\ 2x + y \leq 12 \\ x \geq 0, y \geq 0 \end{cases}$$

Using graphical method, we change inequalities into equalities and the following corners points will occurs: (0,6), (2,5), (5,2), (6,0), and origin (0,0).

Two points (2,5), and (5,2) will be obtained after solving the following two systems respectively $\begin{cases} x+2y=12 \\ x+y=7 \end{cases}$, and $\begin{cases} 2x+y=12 \\ x+y=7 \end{cases}$



To maximize profit, we will substitute the points in the objective function to see which point gives us the maximum profit each day. Critical Points and Income at each are below: (0, 0)

$20(0)+30(0)=\$0,\ (0,6)\ 20(0)+30(6)=\$180,$
$(2,5)\ 20(2)+30(5)=\$190,\ (5,2)\ 20(5)+30(2)=\$160,\ (6,0)\ 20(6)+30(0)=\$120$

The point (2, 5) gives the most profit, and that profit is $190.

Therefore, we conclude that we should manufacture 2 regular devices and 5 premium devices daily to obtain the maximum profit of $190.

The "standard maximization problems" in which the objective function is to be maximized, all constraints are of the form $ax+by\le c$, and all variables are constrained to by non-negative $(x\ge 0, y\ge 0)$.

Application activity 2.3.1

Aline holds two part-time jobs, storekeeper, and Accountant. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at storekeeper, she needs 2 hours of preparation time, and for every hour she works at accountant, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If Aline makes \$40 an hour at Storekeeper, and \$30 an hour at Accountant, how many hours should she work per week at each job to maximize her income?

2.4 End unit assessment

1. An airline must sell at least 25 business class and 90 economy class tickets to earn profit. It makes a profit of320 by selling each business class ticket and a profit of 400 by selling an economy class ticket. No more than 180 passengers can board the plane at a time.

i. How many of economy and business class tickets should be sold by the airline to maximize the profit?

ii. How much maximum profit the airline can earn?

2. A bakery manufacturers two kinds of cookies, chocolate chip, and caramel. The bakery forecasts the demand for at least 80 caramel and 120 chocolate chip cookies daily. Due to the limited ingredients and employees, the bakery can manufacture at most 120 caramel cookies and 140 chocolate chip cookies daily. To be profitable the bakery must sell at least 240 cookies daily. Each chocolate chip cookie sold results in a profit of \$0.75 and each caramel cookie produces \$0.88 profit.

i. How many chocolate chip and caramel cookies should be made daily to maximize the profit?

ii. Compute the maximum revenue that can be generated in a day?

3. A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them A,B and C), it is necessary to buy two additional products, say, X and Y. One unit of product X contains 36 units of A, 3 units of B and 20 units of C. One unit of product Y contains 6 units of A, 12 units of B and 10 units of C. The minimum requirement of A, B and C is 108 units, 36 units and 100 units respectively. Product X costs 20FRW per unit and product Y costs 40 FRW per unit. Formulate the above as a linear Programming problem to minimize the total cost, and solve the problem by using graphic method.

UNIT 3 INTEGRALS

Key unit Competence: Use integration to solve mathematical and financial related problems involving marginal cost, revenues and profits, elasticity of demand, and supply

Introductory activity

If a firm faces the marginal cost schedule given by $MC = 180 + 0.3q^2$ and the marginal revenue schedule given by $MR = 540 - 0.6q^2$ and the total fixed costs are 65,000Frw.

a) Graph MC *and* MR on the same graph
b) Show the positive marginal profit and negative marginal profit
c) Calculate the profit maximization point
d) Calculate total cost and total revenue
e) what is the maximum profit it can make?
f) Find break-even point
g) As an accountant, advise other accountants on how to calculate the maximum profit.

3.1 Indefinite integral

3.1.1 Definition and properties

Learning activity 3.1.1

Perform the following:

i. Find a function $F(x)$ whose derivative is $f(x)=2x$, that is, $F'(x)=f(x)=2x$

ii. Discuss the number of possibilities for $F(x)$ which are there and the relationship among them.

iii. How do functions $F(x)$ differ?

Anti-derivatives

Let $y=f(x)$ be a continuous function of variables. An anti-derivatives of $f(x)$ is any function $F(x)$ such that $F'(x)=f(x)$. A function has infinitely many anti-derivatives, all of them differing by an additive constant. It means that if $F(x)$ is an anti-derivative of $f(x)$, $F(x)+c$ (Where c is an arbitrary constant) is also an anti-derivative of function $f(x)$

Example 1:

Given the function $f(x)=x\ln x-x$

a) Find the derivative of $f(x)$

b) From the answer in (a), deduce the anti-derivative of $g(x)=\ln x$ whose graph passes through point $(e,1)$. Plot the graph of the function $g(x)$ and its anti-derivative on the same rectangular coordinate.

Solution:

a) $f'(x)=(x\ln x-x)'=(x\ln x)'-(x)'=x'\ln x+x(\ln x)'-x'=\ln x$.

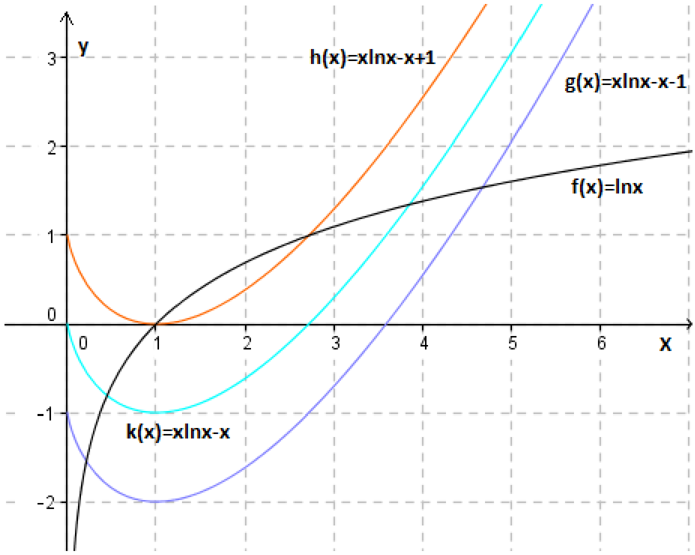
b) The anti-derivatives of $g(x)=\ln x$ are of the type $F(x)=x\ln x-x+c$

$$F(e)=e\ln e-e+c=1\Leftrightarrow c=1$$

Therefore, the required anti-derivative is $F(x)=x\ln x-x+1$ $F(x)=x\ln x-x+1$

The figure below shows function $g(x)=\ln x$ and three of its anti-derivatives.

Figure 1: Graph of the function $f(x) = \ln x$ and 3 of its anti-derivatives



Indefinite integrals

Let $y = f(x)$ be a continuous function of variable x. The indefinite integral of $f(x)$ is the set of all its anti-derivatives. If $F(x)$is any anti-derivative of function and $F(x)$ is any anti-derivative of function $f(x)$, then the indefinite integral of $f(x)$ is denoted and defined as follows:

$\int f(x)dx = F(x) + c$, where c is an arbitrary constant which is called the constant of integration, $\int$ is the sign of integration while $f(x)$is integrand. Note that the integrand is a differential, dx shows that one is integrating with respect to variable x.

Thus, $\int f(x)dx = F(x) + c$ if and only if $[F(x)+c]' = F'(x) = f(x)$

Therefore, the process of finding the indefinite integral of a function is called integration.

Example 2:

Evaluate the following indefinite integrals

a) $\int 7dx$ b) $\int e^t dt$ c) $\int \frac{1}{x}dx$, where $x > 0$ d) $\int (x^3 + 3)dx$

Solution:

a) $\int 7dx = 7x + c$ b) $\int e^t dt = e^t + c$ c) $\int \frac{1}{x}dx = \ln x + c,\ x > 0$

d) $\int (x^3 + 3)dx = \frac{1}{4}x^4 + 3x + c$

Application activity 3.1.1

Evaluate the following integrals

a. $\int 2xdx$ b. $\int 3e^x dx$ c. $I = \int x^2 dx$ d. $\int (3x^3 + 1) dx$

3.1.2 Properties of indefinite integral

Learning activity 3.1.2

Let $f(x) = 5$ and $g(x) = \frac{1}{x}$

a) Determine $\int f(x)dx$ and $\int g(x)dx$

b) Compare $\int (f+g)(x)dx$ and $\int f(x)dx + \int g(x)dx$

c) Compare $\int (f-g)(x)dx$ and $\int f(x)dx - \int g(x)dx$

Let $y = f(x)$ and $y = g(x)$ be continuous functions and k a constant. Integration obeys the following properties/rules:

Rule 1: The integral of the product of a constant by a function is equal to the product of the constant by the integral of the function.

$$\int kf(x) = k\int f(x)dx$$

Example

$$\int 2x^2 dx = 2\int x^2 dx = 2\left(\frac{x^2}{3} + c\right)$$

Rule 2: The integral of a sum or difference of functions equal the sum or difference of the integrals of the function

$$\int [f(x) \pm g(x)]dx = \int f(x)dx + \int g(x)dx$$

Example

$$\int (5e^x - x^{-2} + \frac{3}{x})dx = \int (5e^x)dx - \int (x^{-2})dx + \int \frac{3}{x}dx$$

$$= 5\int e^x dx - \int x^{-2}dx + 3\int \frac{1}{x}dx$$

$$= \left(5e^x + c_1\right) - \left(\frac{x^{-1}}{-1} + c_2\right) + \left(3\ln x + c_3\right)$$

$$= 5e^x + \frac{1}{x} 3\ln x + c$$

Rule 3: The derivative of the indefinite integral is equal to the function to be integrated.

$$\frac{d}{dx}\int f(x)dx = f(x)$$

Note: Integration is a process which is the inverse of differentiation. In differentiation, we are given a function and we are required to find its derivative or differential coeffient. In integration, we find a function whose differential coeffient is given. The process of finding a function is called integration and it reverses the operation of differentiation.

Application activity 3.1.2

1. Evaluate the following integrals

   a) $\int \left(x^3 + 3\sqrt{x} - 7\right)dx$ b) $\int \left(4x - 12x^2 + 8x^6 - 9\right)dx$

2. If $f(x)$ is the total cost function of producing x units of a certain item, then the marginal cost is the derivative with respect to x of the total cost. Given that the marginal cost is $M(x) = 1 + 50x - 4x^2$

   c) Calculate the total cost for 2000Frw each unit.

   d) Represent the graph of $f(x)$ and $M(x)$ on the same diagram

3.1.3 Primitive functions

Learning activity 3.1.3

Consider the following functions $F(x)=\ln(x)$ and $f(x)=\frac{1}{x}$

e) Find $F'(x)$ and $\int f(x)dx$

f) What is the relationship between $F(x)$ and $f(x)$

The function $F(x)$ is said to be a primitive (primitive function, indefinite integral) of the function $f(x)$in the interval (a, b), if the relation $F'(x)=f(x)$ holds for all $x\in(a,b)$. Most of theorems on primitive functions concern their existence, determination and uniqueness. A sufficient condition for existence of a primitive function of a function $f(x)$given on an interval is that $f(x)$is continuous. Any two primitive functions of function given on interval differ by a constant.

Each function $f(x)$which is continuous in (a, b)there exists a primitive. In fact, there exists an infinite number of them. If $F(x)$is a primitive, then all others are of the form $F(x)+c$ where c is an arbitrary constant.

Primitive functions of elementary functions

Given any anti-derivative $F(x)$ of a function $f(x)$, every possible anti-derivative of $f(x)$ can be written in the form of $F(x)+c$,where c is any constant.

Let consider the derivative $(\ln x)'=\frac{1}{x}$, but also $(\ln(-x))'(-1)=\frac{1}{x}$. But $\ln(x)$ and $\ln(-x)$ do not differ just by shifting the constant, so $\frac{1}{x}$ function has two fundamentally different primitive functions, contrary to the statement. How is it possible? So, by reversing the direction of formulas for derivatives of elementary functions the following table of primitive functions displayed.

Function $f(x)$	Primitive function $F(x)$	On interval
$x^k, k \in \mathbb{R} \setminus \{-1\}$	$\frac{x^{k+1}}{k+1}+c$	$]0,\infty[$
$x^k, k \in \mathbb{Z}, k<-1$	$\frac{x^{k+1}}{k+1}+c$	$]0,\infty[$ and $]-\infty,0[$
$x^k, k \in \mathbb{Z}, k>-1$	$\frac{x^{k+1}}{k+1}+c$	$\mathbb{R}$
$x^{-1}=\frac{1}{x}$	$\ln\lvert x\rvert+c$	$]0,\infty[$ and $]-\infty,0[$
e^x	e^x+c	$\mathbb{R}$
e^{mx}	$\frac{e^{mx}}{m}+c$	$\mathbb{R}$
a^{mx}	$\frac{a^{mx}}{m\ln a}+c$	$\mathbb{R}$

Examples :

1. Evaluate $\int 3^{2x}dx$

Solution: $\int 3^{2x}dx=\frac{3^{2x}}{2\ln 3}+c$

2. Evaluate $\int x^2dx$ and the interval of the primitive function

Solution

$$\int x^2dx=\frac{1}{3}x^3+c$$

Application activity 3.1.3

Compute the following indefinite integrals:

a. $\int e^{3x+1}dx$ b. $\int 3^{x}dx$ c. $\int (8-x^{5})dx$

3.2 Techniques of integration

3.2.1 Integration by substitution or change of variable

Learning activity 3.2.1

The given integral $\int e^{5x+2}dx$. By letting $u=5x+2$ and differentiating u with respect to x

a) Deduce dx in function of u

b) Discuss how to determine $\int e^{5x+2}dx$ using expression of u.

Integration by substitution is based on rule of differentiating composite functions. As well as, any given integral is transformed into a simple form of integral using this method of integration by substitution by substituting other variables for the independent variable. When we make a substitution for a function whose derivative is also present in the integrand, the method of integration by substitution is extremely useful. As a result, the function becomes simpler, and the basic integration formulas can be used to integrate the function.

The formula for integration by substitution,

If $F(x)=\int f(x)dx$, the indefinite integral can be obtained by resorting to transformation. If you take $x=g(y)$, then $\int f(x)dx=\int [g(y)]g'(y)dy$. See that $x=g(y)\Rightarrow dx=g'(y)dy$. Referring to the table above of basics formulas of integrations

Example 1

Calculate the following integral $\int (1+5x)^{\frac{1}{2}} dx$

Solution

Let $y = 1+5x$

$dy = 5dx, or\ dx = \frac{1}{5}dy$

Therefore, $\int (1+5x)^{\frac{1}{2}} dx = \frac{1}{5}\int (y)^{1/2} dy = \left(\frac{1}{5}\right)\left(\frac{2}{3}\right)(y)^{3/2} + c$

$= \frac{2}{15} y^{3/2} + c$ replace the value of y

$= \frac{2}{15}(1+5x)^{3/2} + c$

Example 2

Find the value of $F(x) = \int \frac{dx}{(3x-1)^2}$

Solution

Let y $= 3x-1$. So, $dy = 3dx$ or $dx = \frac{1}{3}dy$

Therefore, $F(x) = \frac{1}{3}\int \frac{dy}{y^2} = \frac{1}{3}\left(-\frac{1}{y}\right) + c$ replace the value of y

$= \frac{1}{3(3x+1)} + c$

Example 3

Evaluate $\int \frac{x}{2x^2+3} dx$

Solution

Let $y = 2x^2 + 3$

$$dy = 4xdx, \, or \; dx = \frac{dy}{4x}$$

Hence, $\int \frac{x}{2x^2+3}dx = \int \frac{x}{y} \cdot \frac{dy}{4x} = \frac{1}{4}\int \frac{dy}{y}$

$$= \frac{1}{4}\ln|y| + c = \frac{1}{4}\ln\left(2x^2+3\right) + c$$

Application activity 3.2.1

3. Find $\int (2x+1)^4 dx$
4. Evaluate the following integral

a. $\int \left(x^2+1\right)2xdx$ b. $\int x^2 e^{x^3} dx$ c. $\int (2x+1)e^{x^2+x+2}dx$

3.2.2 Integration by Parts

Learning activity 3.2.2

Look at the function $f(x) = (x-1)e^x$.

5. By using the product rule of differentiation, Differentiate $f(x)$
6. From (1) determine the value of $\int xe^x dx$.
7. Is it true that $\int uvdx = \int udx \int vdx$? Justify your answer

Let us have a short review to the differentiation of product of two functions $u = f(x)$ and $v = f(x)$

$d(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$. From this, we can integrate both sides with respect to x to get

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx \Rightarrow uv = \int vdu + \int udv$$

Therefore, $\int udv = uv - \int vdu$ or $\int vdu = uv - \int udv$ This is the formula for integration by parts.

To apply the integration by parts to a given integral, we must first factor its integrand into two parts. Moreover, an effective strategy is to choose for dv the function which is very easy to integrate and the function u is very easy to differentiate. The following table can be used both values of u and v' *or* dv to apply integral by parts formula

u	v' *or* dv
Logarithmic function	Polynomial function
Polynomial function	Exponential function

Example 1

Find $\ln xdx$Solution

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$, $dv = dx \Rightarrow v = x$

Since, $\int udv = uv - \int vdu$

Then, $\int \ln xdx = x\ln|x| - \int x\frac{dx}{x} = x\ln|x| - \int dx = x\ln|x| - x + c$

Example 2

Find $\int xe^x dx$

Solution

Let $u = x \Rightarrow du = dx$ and $dv = e^x dx \Rightarrow v = e^x$

Then, $\int xe^x dx = xe^x - \int e^x dx$

$= xe^x - e^x + c$

Example 3

Find the value of $F(x)$ for $F(x) = \int xe^x dx$

Solution

Let $x = u(x), v(x) = -e^{-x}$ so that $u'(x) = 1, v'(x) = e^{-x}$

Therefore, $\int uv'dx = uv - \int vu'dx$

$= xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + c = -e^{-x}(1+x) + c$

Then, $F(x) = -e^{-x}(1+x) + c$

Alternative general guidelines for choosing u and dv

i. Let dv be the most complicated portion of the integrand that can be easily integrated

ii. Let u be that portion of the integrand whose derivative du is a simpler function than u itself.

Example

Evaluate the integral $\int x^3\sqrt{4-x^2}\,dx$

Solution

Since both of these are algebraic functions, $x\sqrt{4-x^2}$ is the most complicated part of the integrand that can easily be integrated. Therefore,

$dv = x\sqrt{4-x^2}\,dx$ and $u = x^2 \Rightarrow du = 2xdx$

$$v = \int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int(-2x)\left(4-x^2\right)^{1/2} dx$$

$$\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(4-x^2\right)^{3/2} = -\frac{1}{3}\left(4-x^2\right)^{3/2}$$

$$\int x^3\sqrt{4-x^3}\,dx = uv - \int vdu$$

$$= x^2\left(-\frac{1}{3}\left(4-x^2\right)^{3/2}\right) - \int -\frac{1}{3}\left(4-x^2\right)^{3/2} dx$$

$$= -\frac{x^2}{3}\left(4-x^2\right)^{3/2} - \frac{1}{3}\int\left(4-x^2\right)^{3/2}\left(-2x\right)dx$$

$$= -\frac{x^2}{3}\left(4-x^2\right)^{3/2} - \frac{1}{3}\left(4-x^2\right)^{5/2}\left(\frac{2}{5}\right) + c$$

$$= -\frac{x^2}{3}\left(4-x^2\right)^{3/2} - \frac{2}{15}\left(4-x^2\right)^{5/2} + c$$

Application activity 3.2.2

Use integration by parts method to find

a. $\int x^3 \ln x dx$ b. $\int xe^{3x}dx$ c. $\int xe^{-2x}dx$

3.2.3 Integration by Decomposition/ Simple fraction

Learning activity 3.2.3

Factorize completely the denominator and then decompose the following given fractions into partial fractions

a. $\frac{x-2}{x^2+2x}$ b. $\frac{x}{x^3+3x+2}$ c. $\frac{2}{x^2-4}$

Hence or otherwise find their anti-derivatives

A rational expression is formed when a polynomial is divided by another polynomial. In a proper rational expression, the degree of the numerator is less than the degree of the denominator. In an improper rational expression, the degree of the numerator is greater than or equal to the degree of the denominator. The integration by decomposition known as integration of partial fractions found by firstly decompose a proper fraction into a sum of simpler fractions.

Partial fractions decomposition

CASE 1: The denominator is a product of distinct linear factors.

For each distinct factor $ax+b$ the sum of partial fractions includes a term of the form $\frac{A}{ax+b}$ Example

Rewrite $\frac{x+5}{(x-4)(x-1)}$ as a sum of simpler fractions

Solution

Let write fraction as $\frac{x+5}{(x-4)(x-1)}=\frac{A}{x-4}+\frac{B}{x-1}$

Multiplying both sides by the common denominator to get $x+5=A(x-1)+B(x-4)$

In this linear equation we can replace any value of x to solve for A and B.

Let $x=1$, replace it in equation $1+5=A(1-1)+B(1-4)\Rightarrow 6=-3B\rightarrow B=-2$

Let $x=4$, replace it in equation $4+5=A(4-1)+B(4-4)\Rightarrow 9=3A\rightarrow A=3$

Therefore, the partial fraction decomposition $\frac{x+5}{(x-4)(x-1)}=\frac{3}{x-4}-\frac{2}{x-1}$

Case 2: The denominator is a product of linear factors

For each repeated linear factor $(ax+b)^n$, the sum of partial fractions includes

n terms of the form $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+...+\frac{A_n}{(ax+b)^n}$

Example

Rewrite $\frac{4x}{(x-2)^2}$ as a sum of simpler fractions

Solution

Let us first write the fraction as $\frac{4x}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$

Multiplying both sides by $(x-2)^2$ to get $4x=A(x-2)+B$

Replace $x=2\rightarrow 8=A(2-2)+B,\quad B=8$

Substitute $x=0,\rightarrow 0=(-2)+8\rightarrow A=4$

The partial fraction decomposition is $\frac{4x}{(x-2)^2}=\frac{4}{x-2}+\frac{8}{(x-2)^2}$

Case 3: The denominator has one or more distinct, irreducible quadratic factors

For each distinct factor ax^2+bx+c the sum of partial fractions includes a term $\frac{Ax+B}{ax^2+bx+c}$

Example

Rewrite $\frac{x^2+4x+12}{(x-2)(x^2+4)}$ as a sum of simpler fractions

Solution

The fraction can be written as $\frac{x^2+4x+12}{(x-2)(x^2+4)}=\frac{A}{x-2}+\frac{Bx+c}{x^2+4}$

Multiplying by $(x-2)(x^2+4)$ to get $x^2+4x+12=A(x^2+4)+(Bx+c)(x-2)$

Taking $x=2, 24=A(8)+0\to A=3$

Taking $x=0,\ 12=3(4)+C(-2)\to C=0$

Taking $x=1,\ 17=3(5)+B(-1)\to B=-2$

The partial fraction decomposition is $\frac{x^2+4x+12}{(x-2)(x^2+4)}=\frac{3}{x-2}-\frac{2x}{x^2+4}$

After having the partial fraction decomposition, we apply integration rules to find out the answer to the given indefinite integral

Example 1

Find the indefinite integral of $\int\frac{7x+1}{(x+3)(x-1)}dx$

Solution

Firstly, let partial fraction $\frac{7x+1}{(x+3)(x-1)}=\frac{A}{(x+3)}+\frac{B}{(x-1)}$

$7x+1=A(x-1)+B(+3)$, for $x=1,\to B=2$ and for $x=-3,\to A=5$

Then, $\frac{7x+1}{(x+3)(x-1)} = \frac{5}{(x+3)} + \frac{2}{(x-1)}$

$$\int \frac{7x+1}{(x+3)(x-1)} dx = \int \left(\frac{5}{(x+3)} + \frac{2}{(x-1)} \right) dx$$

$$= \int \left(\frac{5}{(x+3)} \right) dx + \int \left(\frac{2}{(x-1)} \right) dx$$

$$= 5 \ln|x+3| + 2 \ln|x-1| + c$$

Example 2

Evaluate $\int \frac{2x+2}{x^2+2x+1} dx$

Solution

$$x^2+2x+1 = (x+1)(x+1) = (x+1)^2 \Rightarrow \frac{2x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{Ax+A+B}{(x+1)^2}$$

$$\begin{cases} A = 2 \\ A+B = 2 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 0 \end{cases}$$

Therefore, $2\int \frac{dx}{x+1} \rightarrow 2 \ln|x+1| + c$

Case 4: The degree of the numerator is greater or equal to the degree of the denominator, we proceed by long division.

Example

Find $\int \frac{x^2}{x+1} dx$

Solution

Using long division, we get: $\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$

Then, $\int \frac{x^2}{x+1} dx = \int \left(x - 1 + \frac{1}{x+1} \right) dx$

$$= \frac{x}{2} - x + \ln|x+1| + c$$

Application activity 3.2.3

Evaluate the following integrals

1. $\int \frac{2dx}{x^2-1}$ 2. $\int \frac{xdx}{x^2+3x+2}$ 3. $\int \frac{x+3}{x^2-5x+4}dx$

3.3. Definite integral

3.3.1 Definition and properties of definite integrals

Learning activity 3.3.1

Given that $f(x)=x^3+3$ and $g(x)=-2x^2$

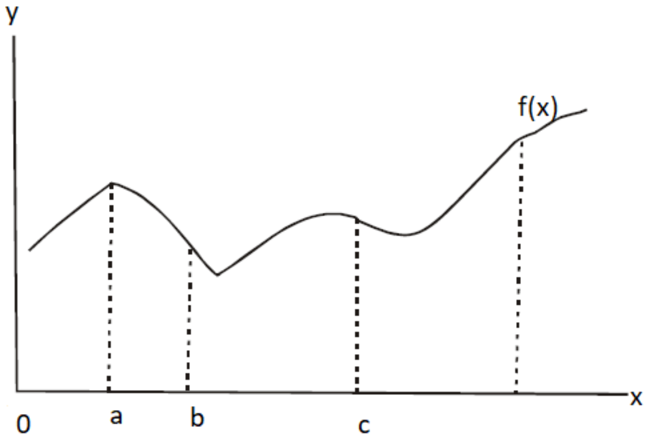
1. Evaluate

a. $\int_1^2 f(x)dx$ b. $\int_1^2 g(x)dx$ c. $\int_1^2 (f+g)(x)dx$

2. compare $\int_1^2 (f+g)(x)dx$ and $\int_1^2 f(x)dx+\int_1^2 g(x)dx$ then justify your answer

3. Generalize results from (2) and conclude on how to find $\int_a^b (f+g)(x)dx$

The definite integral of the function $f(x)$ over the interval $[a,b]$ is expressed symbolically as $\int_a^b f(x)dx$, read as the integral of f with respect to x from a to b. The smaller number a is termed the lower limit and b, the upper limit of the integration.



Note: the indefinite integral $f(x)$ is a function of x while the definite integral $\int_a^b f(x)dx$ is a number. This number depends on the number a as lower limit and b as upper limit of the integration. Definite integral does not rely on such symbol selected to represent independent variable as the form of function not changed. Therefore,

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du = \int_a^b f(y)dy = \dots$$

If $f(x)$ is continuous function on a closed interval $[a,b]$ then fundamental theorem of calculus displayed as $\int_a^b f(x)dx = F(b) - F(a)$. Therefore, if $f(x)$ and $g(x)$ are continuous functions on a closed interval $[a,b]$ then the following rules hold:

1. $\int_a^b 0dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (Permutation of bounds)
3. $\int_a^b [\alpha f(x) \pm \beta g(x)]dx = \alpha \int_a^b f(x)dx \pm \int_a^b g(x)dx,\ \alpha\ and\ \beta \in \mathbb{R}$ (linearity)
4. $\int_a^a f(x)dx = 0$ (bounds are equal)
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$ (Charles relation)

6. $\forall x \in [a,b], f(x) \le g(x) \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$. It follows that $f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0$

and $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ positivity

$$\int_{-a}^{b} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \textit{if } f(x) \textit{ is even function} \\ 0, \text{if } f(x) \textit{ is odd function} \end{cases}$$

Examples

1. Calculate the definite integral $\int_1^2 x^3 dx$

Solution

Fist we calculate $\int_1^2 x^3 dx = \left[\frac{1}{4}x^4\right]_1^2$

Then substitute boundaries $\int_1^2 x^3 dx = \left[\frac{1}{4}x^4\right]_1^2 = \frac{1}{4}\left(2^4 - 1^4\right) = \frac{15}{4}$.

Therefore, $\int_1^2 x^3 dx = \frac{15}{4}$

2. Calculate $\int_0^1 x^2 dx$

Solution

As $\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1$, we have $\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}\left(1^3 - 0^3\right) = \frac{1}{3}$

Application activity 3.3.1

Calculate

1. $\int_0^1 xdx$
2. $\int_1^2 (x^2 - x)dx$
3. $\int_1^2 (3x^2 - 6x)dx$
4. $\int_{-1}^2 (x^3 + 3x - 4)dx$

3.3.2 Techniques of integration of definite integral

Learning activity 3.3.2

1. Consider the continuous function $f(x)=e^{x^2}$ on a closed interval $[a,b]$

i. Let $t=x^2$, determine the value of t when $x=0$ and $x=2$

ii. Determine the value of dx in function of dt

iii. Evaluate the integral $\int_a^b 2xe^{x^2}dx$ using expression of t, consider the results from (i) and (ii)

iv. Explain what happens to the boundaries of the integral when you apply the substitution method

2. 2. Evaluate $\int_1^e x^2 \ln x dx$

There is time that some functions cannot be integrated directly. In that case we have to adopt other techniques in finding the integrals. The fundamental theorem in calculus tells us that computing definite integral of $f(x)$ requires determining its anti-derivative, therefore the techniques used in determining indefinite integrals are also used in computing definite integrals.

a) Integration by substitution

The method in which we change the variable to some other variable is called "**Integration by substitution**".

When definite integral is to be found by substitution then change the lower and upper limits of integration. If substitution is $\varphi(x)$ and lower limit of integration is a and upper limit is b then new lower and upper limits will be φ(a) and φ(b) respectively.

Example:

Evaluate the following definite integrals

i. $\int_0^2 x\sqrt{5-x^2}\,dx$

ii. $\int_0^3 6xe^{x^2+1}dx$

Solution

i. $\int_0^2 x\sqrt{5-x^2}\,dx$

Let $t=5-x^2$, then, $dt=-2xdx$, or $xdx=-\frac{1}{2}dt$

When $when\ x=0,\ t=2, when\ x=2, t=1$

$$\int_0^2 x\sqrt{5-x^2}\,dx=\int_5^1 -\sqrt{t}\frac{dt}{2}=\frac{1}{2}\int_1^5 \sqrt{t}dt$$

$$=\frac{1}{2}\int_1^5 t^{1/2}dt=\frac{1}{2}\left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^5=\frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^5=\frac{1}{2}\times\frac{2}{3}\left[5^{\frac{3}{2}}-1^{\frac{3}{2}}\right]=\frac{1}{3}\left(\sqrt{125}-1\right)$$

ii. $\int_0^3 6xe^{x^2+1}dx$

Let $x^2+1=t$, then $2xdx=dt\ or\ xdx=\frac{1}{2}dt$ when $x=0, t=1$ and when $x=3, t=10$

$$\int_0^3 6xe^{x^2+1}dx=\int_1^{10} 6e^t\frac{dt}{2}=3\int_1^{10} e^t dt=3\left[e^t\right]_1^{10}=3\left(e^{10}-e\right)$$

b) Integration by parts

To compute the definite integral of the form $\int_a^b f(x)g(x)dx$ using integration

by parts, simply set $u=f(x)\ and\ dv=g(x)dx$. Then $du=f'(x)dx$ and $v=G(x)$, anti-derivative of $g(x)$ so that the integration by parts becomes

$$\int_a^b udv=[uv]_a^b-\int_a^b vdu$$

Example :

Evaluate the following definite integral: $\int_0^3 xe^x dx$

Solution

$$I = \int_0^3 xe^x dx$$

Let $\begin{cases} u = x & du = dx \\ dv = e^x dx & v = \int e^x dx = e^x + c \end{cases}$

Applying the integration by parts formula

$I = [uv]_a^b - \int_a^b v du$ to get $I = \left[xe^x\right]_0^3 - \int_0^3 e^x dx = 3e^3 - e^3 + 1 = 2e^3 + 1$

c) Decomposition or simple fractions

The partial fraction decomposition method is useful for integrating proper rational functions. Divide our more complex rational fraction into smaller, and more easily integrated rational functions. When splitting up the rational function, the rules to follow are the same as from the indefinite integral. Here are the steps for evaluating definite integrals using the method of partial fractions :

step 1: Factor the denominator of the integrand

step2: Split the rational function into a sum of partial fractions

step3: Set partial fractions decomposition equal to the original function

step 4: solve for numerators of each of the partial fractions

step 5: Take the definite integral of each of the partial fractions, and sum together

Example :

Evaluate the following integral $\int_2^4 \frac{6}{x^2 - 1} dx$

Solution

$$\frac{6}{x^2 - 1} = \frac{A}{x+1} + \frac{B}{x+1} \rightarrow 6 = A(x-1) + B(x+1)$$

$when\ x = 1, then\ B = 3\ and\ when\ x = -1, A = -3$

This means that $\frac{6}{x^2 - 1} = -\frac{3}{x+1} + \frac{3}{x-1}$

$$\int_2^4 \frac{6}{x^2-1}dx = \int_2^4 \left(\frac{-3}{x+1}+\frac{3}{x-1}\right)dx$$

$$= -\int_2^4 \frac{3}{x+1}dx + \int_2^4 \frac{3}{x-1}dx$$

$$= -3\left[\ln|x+1|\right]_2^4 + 3\left[\ln|x-1|\right]_2^4$$

$$= -3\left(\ln|4+1| - \ln|2+1|\right) + 3\left(\ln|4-1| - \ln|2-1|\right)$$

$$= -3\ln 5 + 3\ln 3 + 3\ln 3 - 3\ln 1 = -3\ln 5 + 6\ln 3$$

Application activity 3.3.2

Evaluate the following integral by using indicated technique

a) $\int_0^1 \ln(1+x)dx$ (Use integration by parts)

b) $\int_1^5 \frac{x+7}{x^3+2x^2}dx$ (use partial fractions for definite integrals)

3.4 Application of integration

3.4.1 Calculation of marginal cost, revenues and profits

Learning activity 3.4.1

Given that the marginal cost (MC) is the rate of change of the total cost (TC) function or $MC = \frac{dTC}{dq}$, at a certain factory, the marginal cost is $3(q-4)^2$ dollars per unit of q when the level of production is q units.

By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units.

a) Cost function and profit function

If the variable under consideration varies continuously, integration allows us to recover the total function from the marginal function. As a result, total functions such as cost, revenue, production, and saving can be derived from them.

The marginal function is obtained by differentiating the total function. Now, when marginal function is given and initial values are given, the total function can be obtained using integration.

If C denotes the total cost and $M(x)=\frac{dC}{dx}$ is the marginal cost, the cost function given by $C=C(x)=\int M(x)dx+k$, where the constant of integration k represents the fixed cost. On the other hand, Profit function is given by

$P(x)=R(x)-C(x)$

Example 1:

The marginal cost function of manufacturing $x\,units$ of a product is $5-16x+3x^2 Rwf$. Find the total cost of producing 5 up to 20 items.

Solution

$$C=\int_5^{20}\left(5-16x+3x^2\right)dx=\left[5x-\frac{16}{2}x^2+x^3\right]_5^{20}=4950Rwf$$

The required cost is 4950RwF

Example 2:

A company has determined that the marginal cost function for a product of a particular commodity is given by $MC=125+10x-\frac{x^2}{9}$ where $C\,Rwf$ is the cost of 9 product $x\,units$ of commodity. If the fixed cost is 25000Rwf. Calculate the cost of producing 15 units.

Solution

$$MC=125+10x-\frac{x^2}{9}$$

$$C=\int MCdx+k,\rightarrow\int\left(125+10x-\frac{x^2}{9}\right)dx+k$$

$= 125x + 5x^2 - \frac{x^3}{27} + k$

Fixed cost is $k = 25000$ then, $C = 125x + 5x^2 - \frac{x^3}{27} + 25000$

When $x = 15$ then, $C = 125(15) + 5(15)^2 - \frac{(15)^3}{27} + 25000$

$C = 1875 + 1125 - 125 + 25000 = 27,875 Rwf$

Therefore, $C = 27,875 Rwf$

Example 3 :

The marginal cost of production of a firm is given by $C'(x) = 5 + 0.13x$, the marginal revenue is given by $R'(x) = 18$ and the fixed cost is 120Rwf. Find the profit function.

Solution

Given $C'(x) = 5 + 0.13x$, $R'(x) = 18$, and fixed cost is 120

$$\int C'(x)dx = \int (5 + 0.13x)dx$$

$$C(x) = 5x + \frac{0.13x^2}{2} + k_1$$

Since fixed cost is $120 \to k_1 = 120$

Therefore, $C(x) = 5x + \frac{0.13x^2}{2} + 120$ also, $R'(x) = 18$

$\int R'(x)dx = \int 18dx \to R(x) = 18x + k_2$

When $x = 0, R = 0 \to k_2 = 0$

Therefore $R(x) = 18x$

Profit function given by $P(x) = R(x) - C(x)$

$$= 18x - 5x - \frac{0.13x^2}{2} - 120$$

Therefore, $P(x) = 13x - 0.065x^2 - 120$

b) Marginal cost and change in total cost

Suppose $C(q)$ represents the cost of producing q items. The derivative, $C'(q)$

is the marginal cost. Since marginal cost $C'(q)$ is the rate of change of the cost function with respect to quantity, by the fundamental theorem, the integral $\int_a^b C'(q)dq$ represent the total change in the cost function between $q=a$ and $q=b$. In other words, the integral gives the amount it costs to increase production from a units to b units.

The cost of producing 0 units is the fixed cost $C(0)$. The area under marginal cost curve between $q=0$ *and* $q=b$ is the total increase in cost between a producing of 0 and a production of b. This is called "Total variable cost". Adding this to fixed cost gives the total cost to produce b units.

Cost to increase production from a units to b units is given $C(b)-C(a)=\int_a^b C'(q)dq$

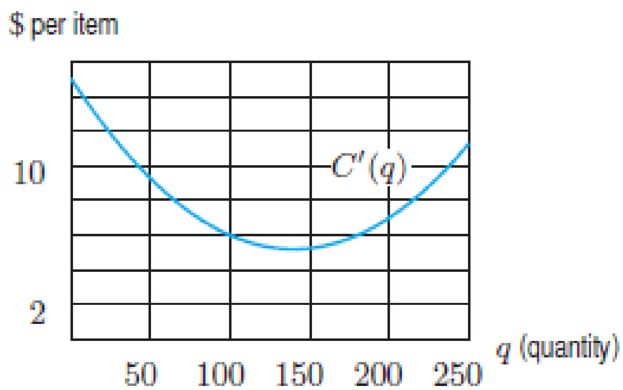
$$Total \text{ variable cos} t \ to \ produce \ b \ units = \int_0^b C'(q)dq$$

$$Total \text{ cos} t \ of \ \ producing \ b \ units = Fixed \text{ cos} t + Total \text{ variable cost}$$

$$= \mathrm{C}(0) + \int_0^b C'(\mathrm{q})\,\mathrm{dq}$$

Example

The following is a marginal cost curve



If the fixed cost is $1000, estimate the total cost of producing 250 items.

Solution

The total cost of production is fixed cost + Variable cost. The variable cost of producing 250 items is represented by the area under the marginal cost curve.

The area in the figure between $q=0$ and $q=250$ is about 20 grid squares.

Each grid square has area (2dollars/ item)(50items) =100 dollars, so

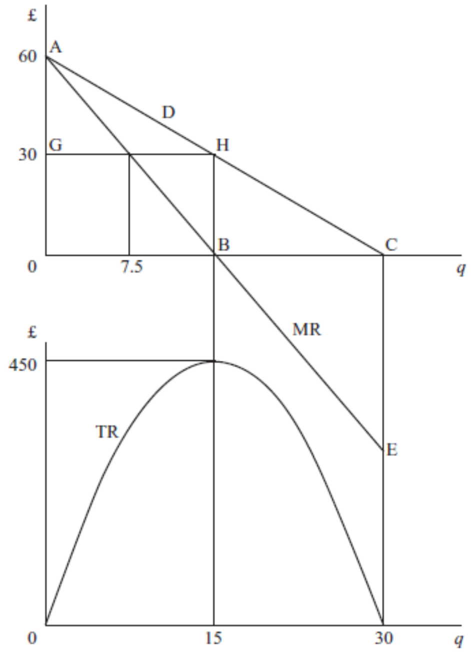
$$\text{Total variable cost} = \int_0^{250} C'(q)dq \approx 20(100) = 2000$$

The total cost to produce 250 items is given by:

$\text{Total cost} = \text{Fixed cost} + \text{Total variable cost}$
$\approx \$1000 + \$2000 = \$3000$

c) Definite integrals of marginal revenue functions

Consider the following phenomenon on MR and TR



This phenomenon is illustrated in the above figure shows the linear demand schedule $p = 60 - 2q$ and the linear marginal revenue schedule $MR = 60 - 4q$.The corresponding total Revenue schedule $TR = 60q - 2q^2$ is shown in the lower part of the diagram. Total revenue is at its maximum when $MR = 60 - 4q = 0$, $q = 15$

Therefore, $p = 60 - 2(15) = 30$ and so the maximum value of TR is $pq = 450$

Given the linear demand and marginal revenue schedules we can see that TR rises from 0 to 450 when q increases from 0 to 15, and then falls back again to zero when q increases from 15 to 30. These changes in TR correspond to the values of the definite integrals over these quantity ranges and are represented by the area between the MR schedule and the quantity axis. When q is 15, TR will be equal to the area 0AB which is

$$\int_0^{15} MRdq = \int_0^{15} (60-4q)dq = \left[60q-2q^2\right]_0^{15} = 900-450$$

The change in TR when q increases from 15 to 30 will be the 'negative' area BCE which lies above the MR schedule and below the quantity axis. This will be equal to

$$\int_{15}^{30} MRdq = \int_{15}^{30} (60-4q)dq = \left[60q-2q^2\right]_{15}^{30}$$

$$= (1800-1800)-(900-450) = -450$$

This checks with our initial assessment. Total revenue rises by 450 and then falls by the same amount. Finally, let us see what happens when we look at the definite integral of the MR function over the entire output range 0–30.

This will be $\int_0^{30} MRdq = \int_0^{30} (60-4q)dq = \left[60q-2q^2\right]_0^{30} = 1800-1800 = 0$

The negative area BCE has exactly cancelled out the positive area 0AB, giving zero TR when q is 30, which is correct.

Application activity 3.4.1

1. Given that the marginal cost MC of an industry $7.5q^2-26q+50$, determine the total cost function
2. If the marginal revenue function $R'(x) = 1500-4x-3x^2$. Find the revenue function and average revenue function
3. Find the total revenue function if the marginal revenue for $x\,units$ is given by $10+3x+x^2$ Hence compute the total revenue given that the output is 2000FRW

3.4.2 Elasticity of demand and supply

Learning activity 3.4.2

The table below shows information about the demand and supply functions for a product. For both functions, q is the quantity and p is the price, in dollars

Table 1

q	0	100	200	300	400	500	600	700
p	70	61	53	46	40	35	31	28

Table 2

q	0	100	200	300	400	500	600	700
p	14	21	28	33	40	47	54	61

a) Among these table which represents demand and which represents supply?

b) From your observation what is price elasticity demand

c) What is the equilibrium price and quantity?

d) Calculate the consumer and producer surplus at equilibrium price

Price Elasticity of demand is a measure of how responsive a commodity's quantity demanded is to price changes. As a result, its measurement is based on comparing the percentage change in price with the resulting percentage change in quantity demanded.

Elasticity of the function $y = f\ (x)$ at a point x is defined as the limiting case of ratio of the relative change in y to the relative change in x .

$$\eta = \frac{E_y}{E_x} = \lim_{\Delta x \to 0} \frac{dy}{y} \times \frac{x}{dx} = \frac{xdy}{ydx} \text{ therefore, } \eta = \frac{xdy}{ydx}$$

Elasticity of demand $\eta_d = \frac{-pdx}{xdp}; \frac{-dp}{p} = \frac{dx}{x}.\frac{1}{\eta_d}$ integrating both side with respect

x yields $-\int \frac{dp}{p} = \frac{1}{\eta_d} \int \frac{dx}{x}$

Equation yields the demand function 'p' as a function of x. The revenue function can be found out by using integration.

Example 1:

Given the elasticity function is $\frac{x}{x-2}$, find the function when $x=6$ and $y=16$

Solution

$$\frac{E_y}{E_x}=\frac{x}{x-2} \Rightarrow \frac{xdy}{ydx}=\frac{x}{x-2}$$

$$\frac{dy}{y}=\frac{x}{x-2}.\frac{dx}{x}$$

$$\int\frac{dy}{y}=\int\frac{dx}{x-2}$$

$$\log y=\log(x-2)+\log k$$

$$y=k(x-2)$$

When $x=6,\ y=16 \quad \Rightarrow 16=k(6-2)$

$k=4$, $\quad y=4(x-2)$

Example 2:

The elasticity of demand with respect to price p for a commodity is

$\eta_d=\frac{p+2p^2}{100-p-p^2}$. Find demand function where price is 5dollars and the demand

is 70 dollars

Solution

$$\eta_d=\frac{p+2p^2}{100-p-p^2}$$

$$\frac{-pdx}{xdp}=\frac{p\left(2p+1\right)}{100-p-p^2}$$

$$\frac{-dx}{x}=\frac{-\left(2p+1\right)}{p^2+p-100}dp$$

$$\int \frac{dx}{x} = \int \frac{2p+1}{p^2+p-100} dp \Rightarrow \log x = \log\left(p^2+p-100\right) + \log k$$

$$x = k\left(p^2+p-100\right)$$

When 70000 and $p = 5000$

$70 = k(25+5-100)$, then $k = -1$

Hence, $x = 100 - p - p^2$ then Revenue is given by $R = px$

$\text{Revenue} = p(100 - p - p^2)$

Application activity 3.4.2

The elasticity of demand with respect to price for a commodity is given by $\frac{4-x}{x}$, where p is price when demand is x. Find the demand function when price is 4 and the demand is 2. Also find the revenue function.

3.4.3 Present, Future Values of an Income Stream and Growth Rates

Learning activity 3.4.3

A company is considering purchasing a new machine for its production floor. The machine costs \$65,000. The company estimates that the additional income from the machine will be a constant \$7000 for the first year, then will increase by \$800 each year after that. In order to buy the machine, the company needs to be convinced that it will pay for itself by the end of 8 years with this additional income. Money can earn 1.7% per year, compounded continuously. Should the company buy the machine?

A basic concept in capital theory is the present or discounted or capital value of a specified sum of money that will be available at a future date. From the compound interest compounding n times per year given by $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$.

If this formula used to find what an account will be worth in the future, $t > 0$ *and* $A(t)$ **is the future value.** If the formula used to find what you need to deposit today to have a certain value P in the future, $t < 0$ *and* $A(t)$ is **called the present value.** Calculus enables us to deal with situations in which deposits are continuously flowing into an account that earns interest. We can use integrals to calculate the present and future value of a continuous income stream as long as we can model the flow of income with a function. The idea here is that each little bit of income in the future needs to be multiplied by the exponential function to bring it back to the present, and then we'll add them all up.

Continuous income stream

Suppose money can earn interest at annual interest rate of r, compounded continuously. Let $F(t)$ be a continuous income function (in dollars per year) that applies between year $0\ and\ year\ T$. Then the present value of that income stream is given by $PV = \int_0^T F(t)e^{-rt}dt$

Example 1

Consider that you have an opportunity to buy a business that will earn \$75,000 per year continuously over the next eight years. Money can earn 2.8% per year, compounded continuously. Is this business worth its purchase price of \$630,000? Justify your answer

Solution

To find the present value of the business, we think of it as an income stream. The function $F(t)$ in this case is a constant \$75,000 dollars per year, so $F(t) = 75,000$. The interest rate is 2.8% and the term interested in 8 years, so, $r = 0.028$ and $T = 8$

$$PV = \int_0^8 75000e^{-0.028t}dt \approx 537,548.75$$

The present value of the business is about \$537,500, which is less than the \$630,000 asking price, so this is not a good deal.

Example 2

Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 6% compounded continuously.

Solution

$S(t) = 1000\ and\ r = 0.06$, we have $Present\ value = \int_0^{20} 1000e^{-0.06t}dt = \$11,647$

We can get the future value B from the present value P, using $B = Pe^{rt}$. So,

$$Future\ value = 11,647e^{0.06(20)} = \$38,669$$

Application of integrals on the population growth rates

The growth of a population is usually modeled by an equation of the form $\frac{dP}{P} = Kdt$ where P represents the number of individuals on a given time t. The constant K is a positive constant when the population grows and negative when the population decreases.

Integration of each side gives: $\int \frac{dP}{P} = \int Kdt$ Which implies that $\ln P = Kt + c$ and $P = ce^{Kt}$

If the initial population at time $t = 0$ is P_0, then $P_0 = ce^0 = c$. Therefore, we have $P = P_0 e^{Kt}$

This means that the variation of a population from P_0 is modelled by an exponential function: $P(t) = P_0 e^{Kt}$ where P_0 is the initial population at time $t = 0$, K the annual growth rate or the annual decay rate.

Example

Consider the population p of a region where there is no immigration or emigration. The rate at which the population is growing is often proportional to the size of the population. This means larger populations grow faster, as we expect since there are more people to have babies.

If the population has a continuous growth rate of 2% per unit time, what is its population at any time t?

Solution

We know that $\frac{dP}{dt} = KP$

$\int \frac{dP}{P} = k \int dt \Rightarrow \ln P = Kt + c$

$P = ce^{kt}$ $for\ k = 0.02$ and has the general solution $P = ce^{0.02t}$

If the initial population at time $t = 0\ is\ P_0$ then $P_0 = ce^{(0.002)(0)} = c$

So, $P_0 = c$ and we have $P = P_0 e^{0.02t}$

Further applications in production and consumption

The supply function or supply curve shows the quantity of a product or service that producers will supply over a period of time at any given price. Both these price-quantity relationships are usually considered as functions of quantity Q

Generally, the demand function $P = D(Q)$ is decreasing, because consumers are likely to buy more of a product at lower prices. Unlike the law of demand, the supply function $P = S(Q)$ increasing, because producers are willing to deliver a greater quantity of a product at higher prices.

The point (Q_0, P_0) where the demand and supply curves intersect is called the market equilibrium point. Consumer surplus CS is thus defined by the integration formula

$$CS = \int_0^{Q_0} D(Q)\,dQ - P_0 Q_0 = \int_0^{Q_0} [D(Q) - P_0]\,dQ$$

A similar analysis shows that producers also gain if they trade their products at the market equilibrium price. Their gain is called producer surplus PS and is given by the equation:

$$PS = P_0 Q_0 - \int_0^{Q_0} S(Q)\,dQ = \int_0^{Q_0} [P_0 - S(Q)]\,dQ$$

Example

For certain product, the demand function is $D(Q) = 1000 - 25Q$ and the supply function is $S(Q) = 100 - Q^2$. Compute the consumer and producer surplus

Solution

First we determine the equilibrium point by equating the demand and supply functions

$$D(Q) = S(Q), \Rightarrow 1000 - 25Q = 100 + Q^2, \Rightarrow Q^2 + 25Q - 900 = 0$$

Solving this quadratic equation $Q_0 = 20$. The market equilibrium price is $P_0 = 500$. The consumer surplus is given by

$$CS = \int_0^{Q_0} [D(Q) - P_0]dQ = \int_0^{20} [1000 - 25Q - 500]dQ = \int_0^{Q_0} [500 - 25Q]dQ$$

$$CS=\left[500Q-\frac{25Q^2}{2}\right]_0^{20}=10000-5000=5000$$

w

The producer surplus PS given by, $PS=\int_0^{Q_0}[P_0-S(Q_0)]dQ=\int_0^{20}\left(500-100-Q^2\right)dQ$

$$PS=\int_0^{20}\left(400-Q^2\right)dQ=\left[400Q-\frac{Q^3}{3}\right]_0^{20}=8000-2667=5333$$

Application activity 3.4.3

An account fetches interest at the rate of 5% per annum compounded continuously an individual deposit 10,000Frw each year in his account. How much will be in the account after 5 years $\left(e^{0.25}=1.284\right)$

3.5 End unit assessment

1. Evaluate the following integrals

a. $\int(10t^{-3}+12t^{-9}+4t^{3})dt$ b. $\int\frac{x^{8}-6x^{5}+4x^{3}-2}{x^{4}}dx$

c. $\int_{0}^{1}\frac{xe^{x}}{(x+1)^{2}}dx$ d. $\int xe^{x}dx$ e. $\int_{1}^{e}x^{3}\log x dx$

2. Discuss how this unit inspired you in relation to learning other subjects or to your future. If no inspiration at all, explain why.

3. The marginal cost function of a commodity is given by $MC=\frac{x}{\sqrt{x^{2}+1600}}$ and the fixed cost is 500Frw. Find the total cost and average cost

4. The given marginal cost function $MR=20-5x+3x^{2}$, find total revenue function

5. The rate of investment is given by $I(t)=6\sqrt{t}$. Calculate the capital growth between the 4th and the 9th years.

6. The given non-linear demand function $p=1800-0.6q^{2}$ and the corresponding marginal revenue function $MR=1,800-1.8q^{2}$. Find

i. The total revenue (TR) when q is 10

ii. The change in TR when q increases from 10 to 20

iii. Consumer surplus when q is 10

UNIT 4 INDEX NUMBERS AND APPLICATIONS

Key unit Competence: Apply index numbers in solving financial related problems, interpreting a value index, and drawing appropriate decisions.

Introductory activity

An industrial worker was earning a salary of 100,000Frw in the year 2000. Today, he earns 1,200,000Frw.

i. Can his standard of living be said to have risen 12 times during this period?

ii. Which statistical measure used to detect changes in a variable (rising or decreasing)

iii. By how much should his salary be raised so that he is as well off as before?

iv. Describe the methods can be used detect the rise of the price

4.1. Introduction to index numbers

4.1.1 Meaning, types and characteristics of index number

Learning activity 4.1.1

If a businessman wants to measure changes in the price level for a specific group of consumers in his region of business.

a) Which statistical measure he can use to detect changes in a variable or group of variables in his business

b) Help him how he can calculate consumer price index for industrial workers, city workers and agricultural workers.

1. Meaning of index number

An index number is a statistical measure used to detect changes in a variable or group of variables. In addition, a single ratio (or percentage) that measures the combined change of several variables between two different times, places, or situations is referred to as an index number. Index numbers are expressed as percentages.

The relative change in price, quantity, or value in comparison to a base period. An index number is used to track changes in raw material prices, employee and customer numbers, annual income and profits.

A simple index is one that is used to measure the relative change in only one variable, such as wages per hour in manufacturing. A composite index is a number that can be used to measure changes in the value of a group of variables such as commodity prices, volume of production in different sectors of an industry, production of various agricultural crops and cost of living.

The index number measures the average change in a group of related variables over two different situations, such as commodity prices, volume of production in various sectors of an industry, production of various agricultural crops, and cost of living. Index numbers measure the changing value of a variable over time in relation to its value at some fixed point in time, the base period, when it is given the value of 100

2. Characteristics of index numbers

Index numbers have the following important characteristics:

- Index numbers are of the type of average that measures the relative changes in the level of a particular phenomenon over time. It is a special type of average that can be used to compare two or more series made up of different types of items or expressed in different units.
- Index numbers are expressed as percentages to show the level of relative change.
- Index numbers are used to calculate relative changes. They assess the relative change in the value of a variable or a group of related variables over time or between locations.
- Index numbers can also be used to quantify changes that are not directly measurable. For example, the cost of living, price level, or business activity in a country are not directly measurable, but relative changes in these activities can be studied by measuring changes in the values of variables/factors that affect these activities.

3. Types of index numbers

a) Consumer price index

A consumer price index (CPI) tracks price changes in a basket of consumer goods and services purchased by households. CPI measures changes in the price level for a specific group of consumers in a given region. CPI can be calculated for industrial workers, city workers and agricultural workers. Consumer price index is given by:

$$cost\ of\ living\ index = \frac{\sum WP}{\sum W} \text{ where, } P = \frac{P_1}{P_0} \times 100 \text{ and } W \text{ are weights}$$

b) The Producer Price Index

The Producer Price Index (PPI) tracks the average change in selling prices received by domestic producers for their output over time. For many products and services, the prices included in the PPI are from the first commercial transaction. The Producer Price Index chart alerts the market to changes in the prices of products leaving the producers. PPIs are available for several manufacturing and service industries' output.

c) Wholesale Price Index number

The Wholesale Price Index (WPI) is the wholesale price of a representative basket of goods. The wholesale price index number reflects the general price level change. It lacks a reference consumer category, unlike the CPI. For example, the WPI with 2011 as a baseline is 156 in March 2014, indicating that the general price level has risen by 56% during this time.

d) Industrial production index

The industrial production index measures the change in the level of industrial production over a given time period across multiple industries. It represents a weighted average of quantity relatives. The index of industrial production

provided by $P = \frac{\sum q_1 w}{\sum w}$

e) Uses of index numbers

The price indices such as CPI and PPI used to:

- Measure the cost of living;
- Measure of inflationary and deflationary tendencies in the economy,
- Means of adjusting income payments: nominal interest rates, wage determination, taxes and other allowances

The quantity/production Indices used to

- Used in national accounts to assess the performance of the economy
- Good indicator of the economic progress taking place in different sectors, regions and countries to facilitate comparisons

Application activity 4.1.1

1. Discuss why index numbers are called as economic barometer
2. Explain the characteristics of index number.

4.1.2. Construction of indices

Learning activity 4.1.2

For the given data of prices in certain country, answer the questions that follows:

Commodity	1998	1999	2000
Cheese(100g)	120	150	156
Egg (per piece)	30	36	33
Potatoes(per kg)	50	60	57

a) How do you describe these prices over the given years?
b) Describe the methods can be used to detect changes in these variable?
c) Find simple aggregative index for the year 1999 over the year 1998
d) Find the simple aggregative index for the year 2000 over the year 1998

The Index I_t for any time period t is given $I_t = \frac{Value\ in\ period\ t}{value\ in\ period\ 0} \times 100$.

The construction of index numbers considers the definition of the purpose of the index, selection of the base period, selection of commodities, obtaining price quotations, choice of an average, selection of weights and then selection of a suitable formula. Price index numbers are used to explain the various methods of index number construction. Price index number construction methods can be divided into three broad categories, as shown below:

Price indices	
Un-weighted index	Weighted index
• Simple aggregative method • Simple average of price relatives method	• Weighted aggregative method • Weighted average of price relatives method

Un-weighted index

Weights are not assigned to the various items used in the calculation of the un-weighted index number. The following are two un-weighted price index numbers:

i. Simple Aggregate Method

This method assumes that different items and their prices are quoted in the same units. All of the items are given equal weight. The following is the formula for a simple aggregative price index:

$$P = \frac{\sum P_t}{\sum P_0} \times 100$$

Where, $\sum P_t$ is the total of current year's prices for the various items.

$\sum P_0$ is the total of base year's prices for the various items.

Example 1:

From the following data compute price index number for the year 2014 taking 2013 as the base year using simple aggregative method:

Commodity	Prices in the year 2013	Prices in the year 2014
A	1	5
B	2	4
C	3	3
D	4	2

Solution

Commodity	Prices in the year 2013	Prices in the year 2014
A	1	5
B	2	4
C	3	3
D	4	2
	$\sum P_0 = 10$	$\sum P_1 = 14$

The price index number is given by $P = \frac{P_t}{P_0} \times 100 = \frac{14}{10} \times 100 = 140$

This price index of 140 indicates that the aggregate of the prices of the given group of commodities increased by 40% between 2013 and 2014. This price index number, calculated using a simple aggregative method, is only of limited utility. The following are the reasons:

a) This method disregards the relative importance of the various commodities used in the calculation.

b) The various items must be expressed in the same unit. In practice, the various items may be expressed in different units.

c) The index number obtained by this method is untrustworthy because it is influenced by the unit in which the prices of various commodities are quoted.

ii. Simple Average of Price Relatives Method

This method is superior to the previous one because it is unaffected by the unit in which the prices of various commodities are quoted. Because the price relatives are pure numbers, they are independent of the original units in which they are quoted. The price index number is defined using price relatives as follows:

$$P = \frac{\sum \frac{P_t}{P_0} \times 100}{N}$$

where P_t and P_0 indicate the price of the i^{th} commodity in the current period and base period respectively. The ratio $\frac{P_t}{P_0} \times 100$ is also referred to as price relative of the commodity and n stands for the number of commodities.

Example1:

Using the data of example 1 the index number using price relative method can be calculated as follows:

Commodity	Prices in the year 2013	Prices in the year 2014	Price relative index $\frac{P_t}{P_0}\times 100$
A	1	5	500
B	2	4	200
C	3	3	100
D	4	2	50
	$\sum P_0 = 10$	$\sum P_1 = 14$	$\sum \frac{P_t}{P_0}\times 100$

$$P = \frac{\sum \frac{P_t}{P_0}\times 100}{N} = \frac{850}{4} = 212.5$$

As a result, the price in 2014 is 112.5% higher than in 2013. The index number, which is based on a simple average of price relatives, is unaffected by the units in which the commodities' prices are quoted. However, this method, like the simple aggregative method, gives equal weight to all items, ignoring their relative importance in the group.

Weighted Index Number

All items or commodities are given rational weights in a weighted index number. These weights indicate the relative importance of the items used in the index calculation. In most cases, the quantity of usage is the most accurate indicator of importance.

i. Weighted Aggregative Price Indices

Weights are assigned to each item in the basket in various ways in weighted aggregative price indices, and the weighted aggregates are also used in various ways to calculate an index. In most cases, the price index number is calculated using the quantity of usage. The two most important methods for calculating weighted price indices are Laspeyre's price index and Paasche's price index. Laspeyre's price index number is a weighted aggregative price index number that uses the quantity from the base year as the weights. It is provided by:

$$P = \frac{\sum P_t q_0}{\sum P_0 q_0} \times 100$$

Example 2: From the following data compute Laspeyre's index number for current year:

Items	Base year		Current year	
	Prices in Frw	Quantity in Kg	Prices in Frw	Quantity in Kg
A	100	6	500	8
B	200	7	400	7
C	300	8	300	6
D	400	9	200	5

Solution

Items	Base year		Current year			
	Prices in Frw (P_0)	Quantity in Kg (q_0)	Prices in Frw (P_1)	Quantity in Kg (q_1)	$(P_t q_0)$	$(P_0 q_0)$
A	100	6	500	8	3000	600
B	200	7	400	7	2800	1400
C	300	8	300	6	2400	2400
D	400	9	200	5	1800	3600
					Total:10,000	Total:8,000

Laspeyre's Price index number is given by $P = \frac{\sum P_t q_0}{\sum P_0 q_0} \times 100 = \frac{10,000}{8,000} \times 100 = 125$

As can be seen here, the value of base period quantities has increased by 25% due to price increases. It is claimed that the price has increased by 25%. The weighted aggregative price index number used by Paasche is the current year's quantity as the weights. It is provided by:

$$P = \frac{\sum P_1 q_1}{P_0 q_1} \times 100$$

In the given **example 2** (above), Paasche's price index number can be calculated as follows:

Items	Base year		Current year			
	Prices in Frw (P_0)	Quantity in Kg (q_0)	Prices in Frw (P_1)	Quantity in Kg (q_1)	(P_1q_1)	(P_0q_1)
A	100	6	500	8	4000	800
B	200	7	400	7	2800	1400
C	300	8	300	6	1800	1800
D	400	9	200	5	1000	2000
					Total:9,600	Total:6,000

Paasche's price index number is given by $P = \frac{\sum P_1q_1}{\sum P_0q_1} \times 100 = \frac{9,600}{6,000} \times 100 = 160$

Paasche's price index of 160 means the price rise of 60 percent using current year quantities as weights.

ii. Weighted Price Relative Method

Under this method price index is constructed on the basis of price relatives and not on the basis of absolute prices. The price index is obtained by taking the average of all weighted price relatives. It is given by

$$P(weighted\ arithmetic\ mean) = \frac{\sum W\left(\frac{P_1}{P_0} \times 100\right)}{\sum W}$$, Where W represents weights

Weights in a weighted price relative index can be determined by the proportion or percentage of total expenditure on them during the base or current period. The base period weight is generally preferred over the current period weight. It is due to the inconvenience of calculating the weight every year.

Example 3: From the following data compute an index number by using weighted average of price relative method:

Items	Base year		Current year
	Prices (P_0)	Quantity (q_0)	Year price (P_1)
A	100	6	500
B	200	7	400
C	300	8	300
D	400	9	200

Solution

Calculation of price index number by weighted average of price relative method using arithmetic mean:

Items	Base year		Current year		$W = P_0 q_0$	$W\left(\frac{P_1}{P_0}\right)\times 100$
	Prices (P_0)	Quantity (q_0)	Year price (P_1)	Relatives $\left(\frac{P_1}{P_0}\right)\times 100$		
A	100	6	500	500	600	300,000
B	200	7	400	200	1400	280,000
C	300	8	300	100	2400	240,000
D	400	9	200	50	3600	180,000
					Total:8,000	Total:1,000,000

$$P(weighted\ arithmetic\ mean) = \frac{\sum W\left(\frac{P_1}{P_0}\times 100\right)}{\sum W} = \frac{1,000,000}{8,000} = 125$$

Application activity 4.1.2

Compute Laspeyre's and Paasche's index numbers for 2000 from the following data

Commodity	Base year		Current year	
	Price	Expenditure	Price	Expenditure
A	50	100	60	180
B	40	120	40	200
C	100	100	120	12
D	20	80	25	100

4.2 End unit assessment

1. Calculate cost of living index number using Family Budget method from the following data (Price in dollars).

Items	Weight	Price in base year	Price in current year
Food	10	150	225
House rent	5	50	150
Clothing	2	30	60
Fuel	3	30	75
Others	5	50	75

2. The given data for base year and current year in the following table

Commodity	Base year		Current year	
	P_0	q_0	P_1	q_1
A	10	5	20	2
B	15	4	25	8
C	40	2	60	6
D	25	3	40	4

i. Calculate the Laspeyre's index

ii. Paasche's index

3. Collect data from the local vegetable market over a week for, at least 10 items. Try to construct the daily price index for the week. What problems do you encounter in applying both methods for the construction of a price index?

UNIT 5 INTRODUCTION TO PROBABILITY

Key unit Competence: Use probability concepts to solve mathematical. Production, financial, and economical related problems and draw appropriate decisions

Introductory activity

The following data were collected by counting the number of Warehouses in use at MAGERWA over a 20 day period. 3 of the 20 days, only 1 Warehouse was used, 5 of the 20 days, 2 Warehouses were used; 8 of the 20 days, 3 Warehouses were used; and 4 of the 20 days, 4 warehouses were used.

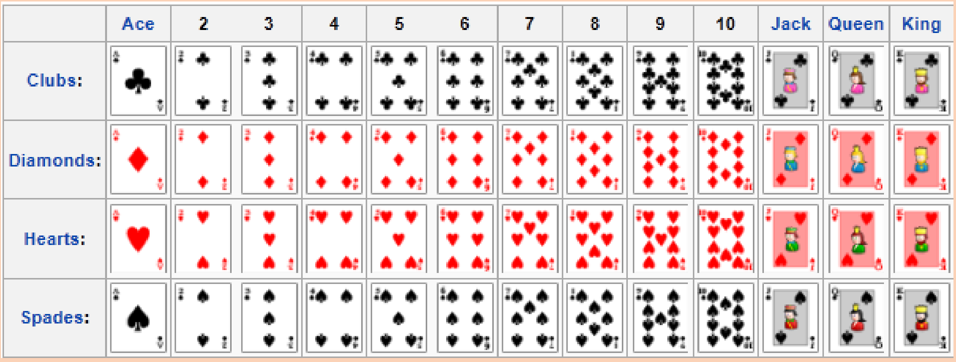
a) Construct a probability distribution for the data
b) Draw a graph for the probability distribution
c) Show that the probability distribution satisfies the required conditions for a discrete probability distribution.
d) On your own side defend why probability is important.

5.1. Key Concepts of probability

5.1.1. *Definitions of probability terminologies*

Learning activity 5.1.1

Consider the deck of 52 playing cards



1. Suppose that you are choosing one card

   a) How many possibilities do you have for the cards to be chosen?

   b) How many possibilities do you have for the kings to be chosen?

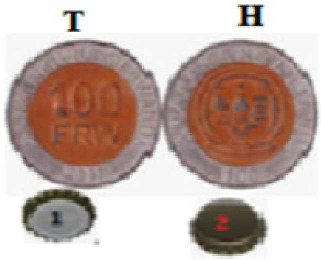
   c) How many possibilities do you have for the aces of hearts to be chosen?

2. If "selecting a queen is an example of event, give other examples of events.

Probability is random variable that can be happen or not, **i.e.** is the chance that something will happen. Using the following examples, we can see how the concept of probability can be illustrated in various contexts

Let us consider a game of playing cards. In a park of deck of 52 playing cards, cards are divided into four suits of 13 cards each. If any player selects a card by random (simple random sampling: chance of picking any card in the park is always equal), then each card has the same chance or same probability of being selected.

Second **example**, assume that a coin is tossed, it may show Head (H-face with logo) or tail(T-face with another symbol), all these are mentioned below:



We cannot say beforehand whether it will show head up or tail up. The result will depend on chance. The same, a card drawn from a well shuffled pack of 52 cards can be **red** or **black**. This also depends on chance. All these phenomena are called probabilistic, means that can be occurred depending on the chance/ uncertainty. The theory of probability is concerned with this type of phenomena.

Probability is a concept which numerically measures the degree of uncertainty and therefore certainty of occurrence of events.

In accounting, the sample of some documents and audited, then results obtained in this sample will be generalized to the whole documents with supporting recommendations to that particular accountant, institutions, and further decisions might be done basing on the findings obtained in the sample.

- **Random experiments and Events**

A **random experiment** is an experiment whose outcome cannot be predicted or determined in advance. These are some example of experiments: Tossing a coin, throwing a dice, selecting a card from a pack of paying cards, etc. In all these cases there are a number of possible results (outcomes) which can occur but there is an uncertainty as to which one of them will actually occur.

Each performance in a random experiment is called a **trial**. The result of a trial in a random experiment is called an **outcome**, an elementary event, or **a sample point**. The totality of all possible outcome (or sample points) of a random experiment constitutes the **sample space** (the set of all possible outcomes of a random experiment) which is denoted by Ω. Sample space may be discrete (single values), or continuous (intervals).

Discrete sample space:

- Firstly, the number of possible outcomes is **finite**.
- Secondly, the number of possible outcomes is **countable infinite**, which means that there is an infinite number of possible outcomes, but the outcomes can be put in a one-to-one correspondence with the positive integers.

Example: If a die is rolled and the number that show up is denoted as

$\Omega = \{1,2,3,\ldots,6\}$ $\Omega = \{1,2,3,\ldots,6\}$.

A die

If a die is rolled until a "6" is obtained, and the number of rolls made before getting first "6" is counted, then we have that write $\Omega = \{1,2,3,\ldots\}$

Continuous sample space

If the sample space contains one or more intervals, the sample space is then **uncountable infinite**.

Example:

A die is rolled until a "6" is obtained and the time needed to get this first "6" is recorded. In this case, we have to denote that $\Omega = \{t \in \mathbb{R}, t > 0\} = (0, \infty)$.

An **event** is a subset of the sample space. The null set ϕ is thus an event known as the **impossible event**. The sample space Ω corresponds to the **sure event.** In particular, every elementary outcome is an event, and so is the sample space itself.

Remarks

- An elementary outcome is sometimes called a **simple event**, whereas a **compound** event is made up of at least two elementary outcomes.
- To be precise we should distinguish between the elementary outcome *w*, which is an element of Ω and the elementary event $\{w\} \subset \Omega$
- The events are denoted by *A*, *B*, *C* and so on.

Example

Consider the experiment that consists in rolling a die and recording the number that shows up. Let A be the event "the even number is shown" and B be the event "the odd number less than 5 is shown". Define the events A and B.

Solution:

We have the sample space $\Omega = \{1,2,3,4,5,6\}$. $A = \{2,4,6\}$ and $B = \{1,3\}$

- Two or more events which have an equal probability (same chance) of occurrence are said to be **equally likely**, i.e. if on taking into account all the conditions, there should be no reason to except any one of the events in preference over the others. Equally likely events may be simple or compound events.
- Two events A and B are said to be **incompatible** (or **mutually exclusive**) if their intersection is empty. We then write that $A \cap B = \varnothing$.
- Two events, A and B are said to be **exhaustive** if they satisfy the condition $A \cup B = \Omega$.
- An event is said to be impossible if it cannot occur.

Example

Consider the experiment that consists in rolling a die and recording the number that shows up. **Solution:**

We have that $\Omega = \{1,2,3,4,5,6\}$.

We define the events: $A = \{1,2,4\}$, $B = \{2,4,6\}$, $C = \{3,5\}$, $D = \{1,2,3,4\}$ and

$E = \{3,4,5,6\}$

We have $A \cup B = \{1,2,4,6\}$, $A \cap B = \{2,4\}$, $A \cap C = \varnothing$, and $D \cup E = \Omega$.

Therefore, A and C are incompatible events. D and E are exhaustive events. Moreover, we may write that $A' = \{3,5,6\}$, where the symbol A' or $\overline{A}$ denotes the complement of the event A.

This suggests the following definition: If E is an event, then E' is the event which occurs when E does not occur. Events E and E' are said to be **complementary events.**

Example of event and sample spaces

Tossing a coin

There are two possible outcomes, you gain Head up or Tail up. Then, $\Omega = \{H,T\}$ throwing a dice and noting the number of its uppermost face. There are 6 possible outcomes: one number from 1 to 6 can be up. Then, $\Omega = \{1,2,3,4,5,6\}$

Two coins are thrown simultaneously. $\Omega = \{HH, HT, TH, TT\}$

Tree coins are thrown simultaneously.

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Two dice are thrown simultaneously

The sample space consists of 36 points: ..... $\Omega = \{(1,1),(1,2),....\}$.

Note: The determination of sample space for some events such as the one for dice thrown simultaneously requires the use of complex reasoning but it can be facilitated by different counting techniques. Morever,the number of possible arrangements for a given set is calculated mathematically, and this process is known as permutation.

Permutation

Permutation is a term that refers to the variety of possible arrangements or orders. The arrangement's order is important when using permutations. Permutations come in three varieties, one without repetition and two with repetition. There is a way you can calculate permutations using a formula

$$P(n,r) = \frac{n!}{(n-r)!} \quad \text{or} \quad {}^{n}P_{r} = \frac{n!}{(n-r)!} \quad \text{or} \quad {}_{n}P_{r} = \frac{n!}{(n-r)!}$$

Where,

- n = total items in the set;
- r = items taken for the permutation;
- "!" denotes taking the factorial

Example 1

(a) $3! = 3 \times 2 \times 1 = 6$

(b) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

(c) $n! = n(n-1)(n-2)(n-3)...3 \times 2 \times 1$

(d) $P\ (10,3) = 10! \div (10-3)! = 10! \div 7! = 10 \times 9 \times 8 = 720$

Example 2

Consider a portfolio manager of a bank screened out 10 companies for a new fund that will consist of 3 stocks. These 3 holdings will not be equal-weighted, which means that ordering will take place. Help the manager to find out the number of ways to order the fund.

Answers

$$P\ (10,3)\ =\ 10!\ \div\ (10-3)!\ =\ 10!\ \div\ 7!\ =\ 10\ \times\ 9\ \times\ 8\ =\ 720$$

Combinations

In contrast to permutations, the combination is a method of choosing things from a collection when the order of the choices is irrelevant. This means that a combination is the choice of r things from a set of n things without replacement and where order does not matter.

$$_{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!} \text{ or } C_{r}^{n} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example

In a bank, the manager organizes election of bank committees consisting with men and women. In how many ways a committee consisting of 5 men and 3 women, can be chosen from 9 men and 12 women?

Answer

Choose 5 men out of 9 men $= {}_{9}C_{5}\ ways\ =\ 126\ ways$

Choose 3 women out of 12 women $=\ {}_{12}C_{3}\ ways\ =\ 220\ ways$

Total number of ways $=\ (126\ \times 220) =\ 27720\ ways$

The committee can be chosen in 27720 ways.

Application activity 5.1.1

1. A box contains 5 red, 3 blue and 2 green pens. If a pen is chosen at random from the box, then which of the following is an impossible event?
   a) Choosing a red pen
   b) Choosing a blue pen
   c) Choosing a yellow pen
   d) None of the above
2. Which of the following are mutually exclusive events when a day of the week is chosen at random?
   a) Choosing a Monday or choosing a Wednesday
   b) Choosing a Saturday or choosing a Sunday
   c) Choosing a weekday or choosing a weekend day
   d) All of the above
3. Two dice are thrown simultaneously and the sum of points is noted, determine the sample space.

5.1.2. Empirical rules, axioms and theorems

Learning activity 5.1.2

Consider the letters of the word "PROBABILITY".

a) How many letters are in this word?
b) How many vowels are in this word?
c) What is the ratio of numbers of vowels to the total number of letters?
d) How many consonants are in this word?
e) What is the ratio of numbers of consonants to the total number of letters?
f) Let A be the set of all vowels and B the set of all consonants. Find

   i. $A \cap B$

   ii. $A \cup B$

   iii. A'

   iv. B'

Empirical rule

The probability of an event $A \subseteq \Omega$, is the real number obtained by applying to A the function P defined by $P(A) = \frac{number\ of\ favour\ outcome}{number\ of\ possible\ outcome} = \frac{cardinal\ of\ A}{cardinal\ of\ \Omega} = \frac{\# A}{\# \Omega}$

Theorem 1:

Suppose that an experiment has only a finite number of equally likely outcomes. If E is an event, then $0 \le P(A) \le 1$.

Notice that,

- If $A = \Omega$, then $P(A) = 1$ and $P(\Omega) = 1$ (the event is certain to occur), and
- If $A = \varnothing$, then $P(A) = 0$ (the event cannot occur).

Example:

A letter is chosen from the letters of the word "**MATHEMATICS**". What is the probability that the letter chosen is an "A"?

Solution:

Since two of the eleven letters are ***A's***, we have two favorable outcomes. There are eleven letters, so we have 11 possible outcomes. Thus, the probability of choosing a letter A is $\frac{2}{11}$.

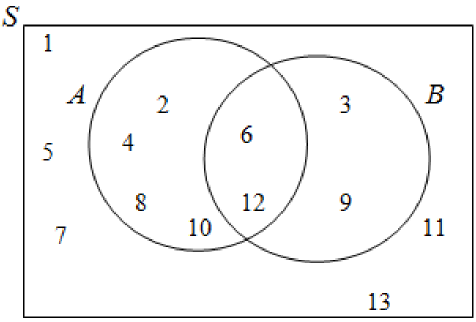
Theorem 2: $P(E) = 1 - P(E')$ where E and E' are complementary events.

Notice that if $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$, where $A_1, A_2, A_3, \ldots, A_n$ are **incompatible events**, then we may write that $P(A) = \sum_{i=1}^{n} P(A_i),\ for\ n = 2, 3, \ldots$

Example

An integer is chosen at random from the set $S = \{x : x \in \mathbb{Z}^+, x < 14\}$. Let A be the event of choosing a multiple of 2 and let B be the event of choosing a multiple of 3. Find $P(A \cup B), P(A \cap B),\ and\ P(A - B)$.

Solution



From the diagram, $\#S = 13$

$A \cup B = \{2,3,4,6,8,9,10,12\} \Rightarrow \#(A \cup B) = 8$, thus $P(A \cup B) = \frac{8}{13}$

$A \cap B = \{6,12\} \Rightarrow \#(A \cap B) = 2$, thus $P(A \cap B) = \frac{2}{13}$

$A - B = \{2,4,8,10\} \Rightarrow \#(A - B) = 4$, thus $P(A - B) = \frac{4}{13}$

Application activity 5.1.2

1. A letter is chosen from the letters of the word "**MATHEMATICS**". What is the probability that the letter chosen is M? and T?
2. An integer is chosen at random from the set $S = \{\text{all positive integers less than } 20\}$. Let A be the event of choosing a multiple of 3 and let B be the event of choosing an odd number. Find

a) $P(A \cup B)$

b) $P(A \cap B)$

c) $P(A - B)$

5.1.3 Additional law of probability, Mutual exclusive, and exhaustive

Learning activity 5.1.3

Consider a machine which manufactures car components. Suppose each component falls into one of four categories: **top quality**, **standard, substandard, reject**

After many samples have been taken and tested, it is found that under certain specific conditions the probability that a component falls into a category is as shown in the following table. The probability of a car component falling into one of four categories.

Category	***Probability***
Top quality	*0.18*
Standard	*0.65*
Substandard	*0.12*
Reject	*0.05*

The four categories cover all possibilities and so the probabilities must sum to 1. If 100 samples are taken, then on average 18 will be top quality, 65 of standard quality, 12 substandard and 5 will be rejected.

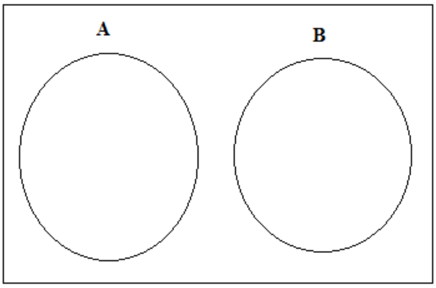
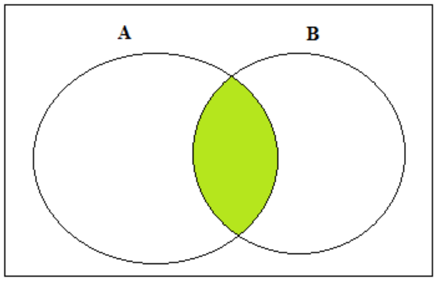
Using the data in table determine the probability that a component selected at random is **either** standard or top quality.

For any event A and B from a sample space E, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;

This is known as the **addition law** of probability from which we deduce that if A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

If E_i and E_j are mutually exclusive we denote this by $E_i \cap E_j = \phi$ that is the compound event $E_i \cap E_j$ is an **impossible event** and so will never occur.

On Venn diagram A and B shown as mutually exclusive (**disjoint sets) events** and shown as no mutually exclusive.

Mutually exclusive (disjoint) events A and B	Non-mutually exclusive events A and B
	

Suppose that $E_1, E_2, E_3, ..., E_n$ are n events and that in a single trial only one of these events can occur. The occurrence of any event, E_i ,excludes the occurrence of all other events. Such events are mutually exclusive.

Generally,

For **mutually exclusive events**, the addition law of probability applies:

$$P(E_1 \; or \; E_2 or \; E_3 \cdots or \; E_n) = P(E_1 \cup E_2 \cup E_3 \cdots \cup E_n) = P\left(\cup_i^n E_i\right)$$

$$= P(E_1) + P(E_1) + P(E_3) + \cdots + P(E_n) = \sum_{i=1}^{n} P(E_i)$$

For **inclusive events** the addition law of probability applies:

$$P(E_1 \; or \; E_2 or \; E_3 \cdots or \; E_n) = P(E_1 \cup E_2 \cup E_3 \cdots \cup E_n) = P\left(\cup_i^n E_i\right)$$

$$= \sum_{i=1}^{n} P(E_i) - \sum_{j>i=1}^{n} P(E_i \cap E_j) + \sum_{k>j>i=1}^{n} P(E_i \cap E_j \cap E_k) + \cdots + (-1)^n P(E_1 \cap E_2 \cap \cdots \cap E_n)$$

The sum of the probability of outcomes

If two events A and B are such that $A \cup B = \Omega$ then $P(A \cup B) = 1$ and then these two events are said to be **exhaustive.** In the sample space, the sum of the probability of outcomes is 1.Generally, given a finite sample space, $\Omega = \{a_1, a_2, a_3, ..., a_n\}$, the finite probability obtained by assigning to each point $a_i \in \Omega$ a real number P_i , called the probability of a_i , satisfying the following:

a) $P_i \geq 0$ for all integers i, $1 \leq i \leq n$;

b) $\sum_{i=1}^{n} P_i = 1$.

Examples:

1. Suppose a student is selected at random from 100 students where 30 are taking Auditing, 20 are taking Taxation, and 10 are taking Auditing and Taxation. Find the probability that the student is taking Auditing or Taxation.

Solution

Let M be the event "students taking Auditing" and C the event "students taking Taxation" then, $P(M)=\frac{30}{100}=\frac{3}{10}, P(C)=\frac{20}{100}=\frac{1}{5}$ and $P(M\cap C)=\frac{10}{100}=\frac{1}{10}$.

Thus, by the addition principle,

$$P(M\ or\ C)=P(M\cup C)=P(M)+P(C)-P(M\cap C)=\frac{3}{10}+\frac{1}{5}-\frac{1}{10}=\frac{4}{10}=\frac{2}{5}.$$

2. In a competition prepared by Rwanda revenue in which there are no dead heats, the probability that John wins is 0.3, the probability that Mike wins is 0.2 and the probability that Putin wins is 0.4. Find the probability that :

a) John or Mike wins

b) John or Putin or Mike wins,

c) Someone else wins.

Solution:

Since only one person wins, the events are mutually exclusive.

a) $P(John\ or\ Mike\ win) = 0.3 + 0.2 = 0.5$

b) $P(\ John\ or\ Putin\ or\ Mike\ win) = 0.3 + 0.2 + 0.4 = 0.9$

c) $P(Someone\ else\ wins) = 1 - 0.9 = 0.1$

3. Machines A and B make components. Machine A makes 60% of the Components. The probability that a component is acceptable is 0.93 when made by machine A and 0.95 when made by machine B. A component is picked at random. Calculate the probability that it is:

a) Made by machine A and is acceptable.

b) Made by machine B and is acceptable.

c) Acceptable.

Solution:

a) 60% of the components are made by machine A and 93% of these are acceptable.

$$P(\textit{component is made by machine A and is acceptable}) = \frac{60}{100} \times \frac{93}{100} = 0.60 \times 0.93 = 0.558$$

b) We know that 40% of the components are made by machine B and 95% of these are acceptable.

$$P(\textit{component is made by machine } \mathrm{B} \textit{ and is acceptable}) = \frac{40}{100} \times \frac{95}{100} = 0.40 \times 0.95 = 0.38$$

c) Note that the events described in (a) and (b) are mutually exclusive and so the addition law can be applied.

P(component is acceptable) = *P*(component is made by machine A and is acceptable)+*P*(component is made by machine B and is acceptable) = 0.558 + 0.38 = 0.938

Application activity 5.1.3

1. A fair die is rolled, what is the probability of getting an even number or prime number?

2. Events A and B are such that they are both mutually exclusive and exhaustive. Find the relation between these two events.

3. In a class of a certain school, there are 12 girls and 20 boys. If a teacher want to choose one student to answer the asked question

   a) What is the probability that the chosen student is a girl?

   b) What is the probability that the chosen student is a boy?

   c) If teacher doesn't care on the gender, what is the probability of choosing any student?

4. On New Year's Eve, the probability of a person driving while intoxicated is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving accident while intoxicated is 0.06.

What is the probability of a person driving while intoxicated or having a driving accident?

5.1.4 Independence, Dependence and conditional probability

Learning activity 5.1.4

Suppose that you have a deck of cards; then draw a card from that deck, not replacing it, and then draw a second card.

a) What is the sample space for each event?

b) Suppose you select successively two cards, what is the probability of selecting two red cards?

c) Explain if there is any relationship (Independence or dependence) between those two events considering the sample space. Does the selection of the first card affect the selection of the second card?

a) Independent events

If probability of event *B* is not affected by the occurrence of event *A*, events *A* and *B* are said to be **independent** and $P(A \cap B) = P(A) \times P(B)$

This rule is the simplest form of the **multiplication law** of probability.

Example

A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

Solution

Let A be the event: "a 4 is obtained on the first throw", then $P(A) = \frac{1}{6}$. That is

$A = \{4\}$

B be the event: "an odd number is obtained on the second throw". That is

$B = \{1, 3, 5\}$

Since the result on the second throw is not affected by the result on the first throw, *A* and *B* are independent events.

There are 3 odd numbers, then

$$P(B) = \frac{3}{6} = \frac{1}{2}$$

Therefore,

$$P(A\cap B)=P(A)P(B)$$
$$=\frac{1}{6}\times\frac{1}{2}$$
$$=\frac{1}{12}$$

Example

A factory runs two machines. The first machine operates for 80% of the time while the second machine operates for 60% of the time and at least one machine operates for 92% of the time. Do these two machines operate independently?

Solution

Let the first machine be M_1 and the second machine be M_2, then

$P(M_1)=80\%=0.8$, $P(M_2)=60\%=0.6$ and $P(M_1\cup M_2)=92\%=0.92$

Now,

$$P(M_1\cup M_2)=P(M_1)+P(M_2)-P(M_1\cap M_2)$$

$$\begin{aligned}P(M_1\cap M_2)&=P(M_1)+P(M_2)-P(M_1\cup M_2)\\&=0.8+0.6-0.92\\&=0.48\\&=0.8\times 0.6\\&=P(M_1)\times P(M_2)\end{aligned}$$

Thus, the two machines operate independently.

Example

A coin is weighted so that heads is three times as likely to appear as tails. Find $P(H)$ and $P(T)$

Solution

Let $P(T)=p_1$, then $P(H)=3p_1$.

But $P(H)+P(T)=1$

Therefore $p_1+3p_1=1\Leftrightarrow 4p_1=1\Rightarrow p_1=\frac{1}{4}$

Thus, $P(H)=\frac{3}{4}$ and $P(T)=\frac{1}{4}$.

b) Dependent events and conditional probability

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be ***dependent***.

Suppose A is an event in a sample space S with $P(A) > 0$. The probability that an event B occurs once A has occurred, written as $P(B|A)$ is called the conditional probability of B given A and is defined as $P(B|A) = \frac{P(A \cap B)}{P(A)}$

From this result, we have general statement of the **multiplication law**:

$$P(A \cap B) = P(A) \times P(B|A)$$

This shows us that the probability that two events will both occur is the product of the probability that one will occur and the conditional probability that the other will occur given that the first has occurred. We can also write $P(A \cap B) = P(B) \times P(A|B)$.

Notice:

If A and B are **independent**, then the probability of B is not affected by the occurrence of A and so $P(B|A) = P(B)$ giving $P(A \cap B) = P(A) \times P(B)$.

Examples:

1. Suppose a card is drawn from a deck and not replaced, and then the second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Solution:

The probability of selecting an ace on the first draw is $\frac{4}{52}$. But since that card is not replaced, the probability of selecting a king on the second card is $\frac{4}{51}$, since there are 51 cards remaining.

The outcomes of the first draw has affected the outcome of the second. By multiplication rule , the probability of both events occurring is :

$$\frac{4}{52} \times \frac{4}{51} = \frac{16}{2652} = \frac{4}{663} = 0.006.$$

2. A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.

Solution

Let A be the event: "the number is a 4", then $A=\{4\}$

B be the event: "the number is greater than 2", then $B=\{3,4,5,6\}$ and $P(B)=\frac{4}{6}=\frac{2}{3}$

But $A\cap B=\{4\}$ and $P(A\cap B)=\frac{1}{6}$

Therefore,

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

$$P(A|B)=\frac{\frac{1}{6}}{\frac{2}{3}}$$

$$P(A|B)=\frac{1}{6}\times\frac{3}{2}$$
$$=\frac{1}{4}$$

3. At a middle school, 18% of all students performed competition on Taxation and auditing, and 32% of all students performed competition on Taxation . What is the probability that a student who performed competition on Taxation also performed auditing?

Solution

Let A be a set of students who performed competition on Taxation and B a set of students who auditing then the set of students who performed both competitions is $A\cap B$. We have $P(A)=32\%=0.32$, $P(A\cap B)=18\%=0.18$. We need the probability of B known that A has occurred.

Therefore,

$$P(B|A)=\frac{P(A\cap B)}{P(A)}$$
$$=\frac{0.18}{0.32}$$
$$=0.5625$$
$$=56\%$$

Contingency table

Contingency table (or **Two-Way table**) provides a different way of calculating probabilities. It helps in determining conditional probabilities quite easily. The table displays sample values in relation to two different variables that may be dependent or contingent on one another.

Below, the contingency table shows the favorite leisure activities for 50 adults, 20 men and 30 women. Because entries in the table are frequency counts, the table is a **frequency table.**

	Dance	**Sports**	**TV**	**Total**
Men	2	10	8	20
Women	16	6	8	30
Total	18	16	16	50

Entries in the total row and total column are called **marginal frequencies** or the **marginal distribution**. Entries in the body of the table are called **joint frequencies**.

Example

Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Car phone user	25	280	305
Not a car phone user	45	405	450
Total	70	685	755

Calculate the following probabilities using the table:

a) P(person is a car phone user)
b) P(person had no violation in the last year)
c) P(person had no violation in the last year AND was a car phone user)
d) P(person is a car phone user OR person had no violation in the last year)
e) P(person is a car phone user GIVEN person had a violation in the last year)

f) P(person had no violation last year GIVEN person was not a car phone user)

Solution

a) $P(\text{person is a car phone user}) = \frac{\text{number of car phone users}}{\text{total number in study}} = \frac{305}{755}$

b) $P(\text{person had no violation in the last year}) = \frac{\text{number that had no violation}}{\text{total number in study}} = \frac{685}{755}$

c) $P(\text{person had no violation in the last year AND was a car phone user}) = \frac{280}{755}$

d) $P(\text{person is a car phone user OR person had no violation in the last year})$

$$= \left(\frac{305}{755} + \frac{685}{755}\right) - \frac{280}{755} = \frac{710}{755}$$

e) The sample space is reduced to the number of persons who had a violation. Then

$P(\text{person is a car phone user GIVEN person had a violation in the last year}) = \frac{25}{70}$

f) The sample space is reduced to the number of persons who were not car phone users. Then

$P(\text{person had no violation last year GIVEN person was not a car phone user}) = \frac{405}{450}$

Application activity 5.1.4

1. A dresser drawer contains one pair of socks with each of the following colors: blue, brown, red, white and black. Each pair is folded together in a matching set. You reach into the sock drawer and choose a pair of socks without looking. You replace this pair and then choose another pair of socks. What is the probability that you will choose the red pair of socks both times?

2. A coin is tossed and a single 6-sided die is rolled. Find the probability of landing on the head side of the coin and rolling a 3 on the die.

3. A husband and a wife appear in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that: both of them will be selected, only one of them will be selected, none of them will be selected?

4. The world-wide Insurance Company found that 53% of the residents of a city had home owner's Insurance with its company of the clients, 27% also had automobile Insurance with the company. If a resident is selected at random, find the probability that the resident has both home owner's and automobile Insurance with the world wide Insurance Company.

5. Calculate the probability of a 6 being rolled by a die if it is already known that the result is even.

6. A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

5.1.5. Tree diagram, Bayes theorem and its applications

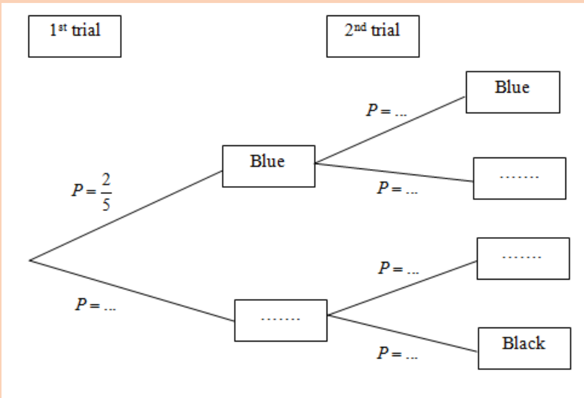
Learning activity 5.1.5

1. A box contains 4 blue pens and 6 black pens. One pen is drawn at random, its color is noted and the pen is replaced in the box. A pen is again drawn from the box and its color is noted.

   a) For the 1st trial, what is the probability of choosing a blue pen and probability of choosing a black pen?

   b) For the 2nd trial, what is the probability of choosing a blue pen and probability of choosing a black pen? Remember that after the 1st trial the pen is replaced in the box.

2. In the following figure complete the missing colours and probabilities



3. Suppose that entire output of a factory is produced on three machines. Let B_1 denote the event that a randomly chosen item was made by machine 1, B_2 denote the event that a randomly chosen item was made by machine 2 and B_3 denote the event that a randomly chosen item was made by machine 3. Let A denote the event that a randomly chosen item is defective.

a) Tree diagram

A tree diagram is a tool in the fields of probability, and statistics that helps calculate the number of possible outcomes of an event or problem, and to cite those potential outcomes in an organized way. It can be used to show the probabilities of certain **outcomes** occurring when two or more **trials** take place in succession. The **outcome** is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring. For each **trial** the number of branches is equal to the number of possible outcomes of that trial. In the diagram there are two possible outcomes, *A* and *B*, of each trial.

Successive trials are events which are performed one after the other; all of which are mutually exclusive.

Examples:

1. A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Find the probability the ball drawn will be

a) Red followed by green,

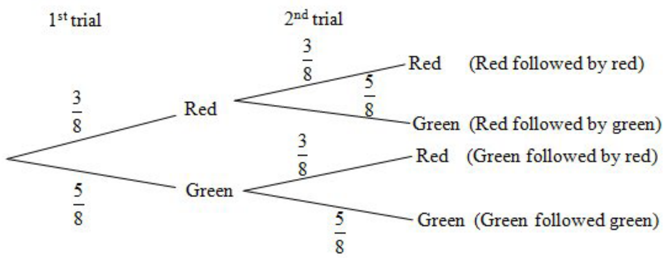
b) Red and green in any order,

c) Of the same color.

Solution

Since there are 3 red balls and 5 green balls, for the 1st trial, the probability of choosing a red ball is $\frac{3}{8}$ and probability of choosing a green ball is $\frac{5}{8}$ and since after the 1st trial the ball is replaced in the bag, for the second trial the probabilities are the same as in the first trial.

Draw a tree diagram showing the probabilities of each outcome of the two trials.



a) $P(\text{Red followed by green}) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$

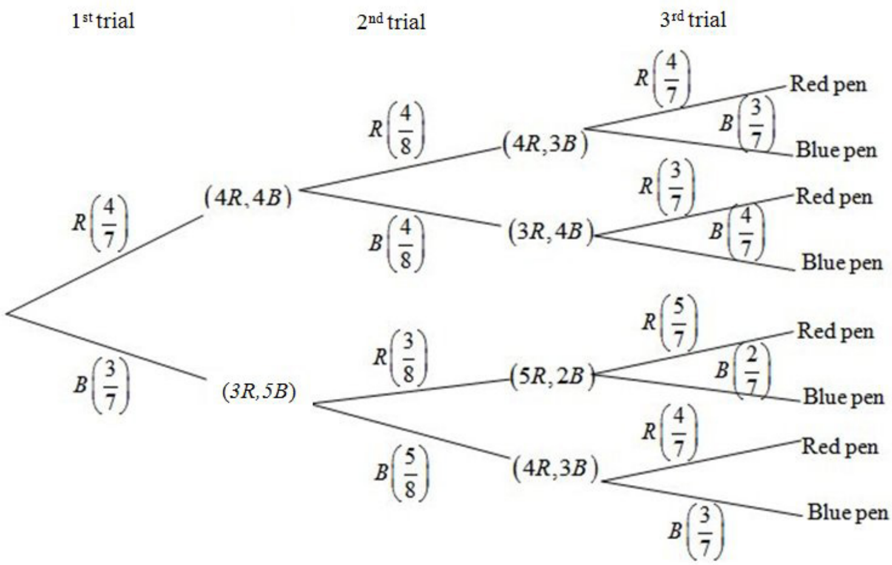
b) $P(\text{Red and green in any order}) = \frac{3}{8}\times\frac{5}{8}+\frac{5}{8}\times\frac{3}{8}=\frac{15}{32}$

c) $P(\text{both of the same colors}) = \frac{3}{8}\times\frac{3}{8}+\frac{5}{8}\times\frac{5}{8}=\frac{17}{32}$

2. A bag (1) contains 4 red pens and 3 blue pens. Another bag (2) contains 3 red pens and 4 blue pens. A pen is taken from the first bag (1) and placed into the second bag (2). The second bag (2) is shaken and a pen is taken from it and placed in the first bag (1). If now a pen is taken from the first bag, use the tree diagram to find the probability that it is a red pen.

Solution

Tree diagram is given below:



From tree diagram, the probability to have a red pen is

$$P(R)=\frac{4}{7}\times\frac{4}{8}\times\frac{4}{7}+\frac{4}{7}\times\frac{4}{8}\times\frac{3}{7}+\frac{3}{7}\times\frac{3}{8}\times\frac{5}{7}+\frac{3}{7}\times\frac{5}{8}\times\frac{4}{7}$$

$$=\frac{64}{392}+\frac{48}{392}+\frac{45}{392}+\frac{60}{392}$$

$$=\frac{31}{56}$$

b) Bayes theorem and its applications

Let $B_1, B_2, B_3, \ldots, B_n$ be incompatible and exhaustive events and let A be an arbitrary event.

We have: $P(B_i \mid A) = \dfrac{P(B_i \cap A)}{P(A)} = \dfrac{P(A \mid B_i)P(B_i)}{\sum_{i=1}^{n} P(A \mid B_i)P(B_i)}$, This formula is called Bayes' formula.

Remark: We also have (Bayes' rule): $P(B \mid A) = \dfrac{P(A \mid B)P(B)}{P(A)}$

Examples:

1. Suppose that machines M_1, M_2, and M_3 produce, respectively, 500, 1000, and 1500 parts per day, of which 5%, 6%, and 7% are defective. A part produced by one of these machines is taken at random, at the end of a given workday, and it is found to be defective. What is the probability that it was produced by machine M_3?

Solution

Let A_i be the event "the part taken at random was produced by machine M_i," for $i = 1, 2, 3$; and let D be "the part taken at random is defective." Using Bayes' formula, we seek

$$P(A_3 \mid D) = \frac{P(D \mid A_3)P(A_3)}{\sum_{i=1}^{3} P(D \mid A_i)P(A_i)}$$

$$= \frac{(0.07)\left(\frac{1500}{3000}\right)}{(0.05)\left(\frac{1}{6}\right) + (0.06)\left(\frac{1}{3}\right) + (0.07)\left(\frac{1}{2}\right)}$$

$$= \frac{105}{190}$$

$$= \frac{21}{38}$$

2. Two machines A and B produce 60% and 40% respectively of total output of a factory. Of the parts produced by machine A, 3% are defective and of the parts produced by machine B, 5% are defective. A part is selected at random from a day's production and found to be defective. What is the probability that it came from machine A?

Solution

Let E be the event that the part came from machine A , C be the event that the part came from machine B, and D be the event that the part is defective.

We require $P(E \cap D)$.Now, $P(E) \times P(D|E) = 0.6 \times 0.03 = 0.018$ and

$P(D) = P(E \cap D) + P(C \cap D) = 0.018 + 0.4 \times 0.05 = 0.038$

Therefore, the required probability is $\frac{0.018}{0.038} = \frac{9}{19}$.

Application activity 5.1.5

1. Calculate the probability of three coins landing on: Three heads.
2. A class consists of six girls and 10 boys. If a committee of three is chosen at random, find the probability of: **a)** Three boys being chosen, **b)** exactly two boys and a girl being chosen. Exactly two girls and a boy being chosen, **d)** three girls being chosen.
3. A bag contains 7 discs, 2 of which are red and 5 are green. Two discs are removed at random and their colours noted. The first disk is not replaced before the second is selected. Find the probability that the discs will be

   a) both red, **b)** of different colours, **c)** the same colours.
4. Three discs are chosen at random, and without replacement, from a bag containing 3 red, 8 blue and 7 white discs. Find the probability that the discs chosen will be

   a) all red   b) all blue   c) one of each colour.
5. 20% of a company's employees are engineers and 20% are economists. 75% of the engineers and 50% of the economists hold a managerial position, while only 20% of non-engineers and non-economists have a similar position. What is the probability that an employee selected at random will be both an engineer and a manager?
6. The probability of having an accident in a factory that triggers an alarm is 0.1. The probability of its sounding after the event of an incident is 0.97 and the probability of it sounding after no incident has occurred is 0.02. In an event where the alarm has been triggered, what is the probability that there has been no accident?

5.2. Probability distributions

5.2.1 Discrete probability distribution

Learning activity 5.2.1

Learning activity 5.2.1

In City of Kigali, data were collected on the number of people who had applied to buy shares in the Rwanda Stock Exchange. People bought different shares although some withdrew their requests. The table below shows the collected data.

Number of people	Number of shares bought
1,218	0
32,379	1
37,961	2
19,387	3
7,714	4
2,842	5
Total 101,501	

Develop the probability distribution of the random variable defined as the number of shares bought per person.

Probability Distribution: The values a random variable can assume and the corresponding probabilities of each. Probability distributions are related to frequency distributions.

For a **discrete random variable x,** the probability distribution is defined by a probability function, denoted by f(x). The probability function gives the probability for each value of the random variable. Then X is a **discrete random variable** if $\sum_{i=1}^{n} p_i = 1 \ or \ \sum_{allx} p(X = x) = 1$

We always denote a random variable by a capital letter X,Y,Z….and the particular values by lower case letter x,y,z….

A discrete Variable is a variable that can take on a finite or countable number of distinct values. It is a variable that is characterized by gaps in the values it can take.

Examples

- number of heads observed in an experiment that flips a coin 10 times,
- number of times a student visits the library;
- number of children in class

A **discrete probability distribution** consists of the values a random variable can assume and the corresponding probabilities of the values. The probabilities are determined theoretically or by observation.

Two Requirements for a Probability Distribution are

- The probability of each event in the sample space must be between or equal to 0 and 1. That is, $0 \le P(X) \le 1$
- The sum of the probabilities of all the events in the sample space must equal 1; that is, $\sum P(X) = 1$

If these two conditions aren't met, then the function isn't a probability function. Given a random variable X, we usually are interested in the probability that X takes one particular value x of its range. We denote this probability by $P(X = x)$.

A **probability function** is a function which assigns probabilities to the values of a random variable.

Random Variable: Is a variable whose values are determined by chance.

Probability density function (pdf) for discrete random variable

This is the function that is responsible for allocating probabilities. It is written as

$$f(x_i) = P(X = x_i) \quad with \quad 1 \le i \le n.$$

x_1	$f(x_1)=P_1$
x_2	$f(x_2)=P_2$
x_3	$f(x_3)=P_3$
……	…….
x_n	$f(x_n)=P_n$

$$0 \le f(x_i) \le 1 \text{ and } \sum_{i=1}^{n} f(x_i)=1$$

Example: The following table gives the pdf of X

I	1	2	3	4
x_i	-3	-2	1	4
$P(X=x_i)$	0.1	0.2	0.3	0.4

Mean, Variance, and Standard Deviation

Expectation or Mean $(\mu = E(X))$

The expected value of a discrete random variable of a probability distribution is the theoretical average or mean of the variable.

The formula is $E(X)=\sum xP(X=x)$ *or* $E(X)=\sum_{i=1}^{n} x_i p_i$ *or* $E(X)=\sum_{i=1}^{n} x_i f(x_i)$

Theorems:

1. Let's consider an random variable X and a real number k:

$$E(kX)=kE(X)$$
$$E(X+k)=E(X)+k$$

2. Let consider X and Y, two random variables:

$$E(X+Y)=E(X)+E(Y)$$

Examples:

1. A random variable has pdf as shown. Find E(X).

X	-2	-1	0	1	2
P(X=x)	0.3	0.1	0.15	0.4	0.05

Solution

$$E(X)=\sum_{all\,x} xP(X=x)=(-2\times 0.3)+(-1\times 0.1)+(0\times 0.15)+(1\times 0.4)+(2\times 0.05)=-0.2$$

- **The expectation of any function of X, E[g(x)]**

Let X be a random variable and probability density function $f(x)=P(X=x)$.

Let also g(x) be a function of the random variable X. Then , the expected value of g(x) written E[g(x)] is given by: $E[g(x)]=\sum_{all\ x} g(x)P(X=x)$

2. The random variable X has probability density function

P(X=x) $P(x_1)=0.1,\quad P(x_2)=0.6,\quad and\ P(x_3)=0.3$ for x=1, 2, 3.

Compute: a) $E(3)$ b) $E(x)$ c) $E(5x)$ d) $E(5x+3)$

Solution

a) $$E(X)=\sum_{all\ x} x\,P(X=x)$$

But since x=3, we have,

$$E(X)=\sum_{all\,x} 3P(X=x)=3\sum_{all\,x} P(X=x)=3(0.1+0.6+0.3)=3$$

b) $E(X)=\sum_{all\ x} xP(X=x)=(1\times 0.1)+(2\times 0.6)+(3\times 0.3)=2.2$

c) $E(5X)=\sum_{all\ x} 5xP(X=x)=(5\times 1\times 0.01)+(5\times 2\times 0.6)+(5\times 3\times 0.3)=11$

d) $E(5X+3)=\sum_{all\ x}(5x+3)P(X=x)=8\times 0.1+13\times 0.6+18\times 0.3=14$

- **Variance (Var X) or σ^2 and standard deviation σ_X**

The variance of X, written Var (X) is given by: $Var(X)=\sigma^2=E(X^2)-\mu^2$

Hence, if X is a random variable having the probability density function $f(x)=P(X=x), x\in\mathbb{R}$ $f(x)=P(X=x), x\in R$ and the $E(X)=\mu$ where μ is constant. Then;

Thus, the standard deviation is $\sigma_x=\sqrt{\sigma^2}=\sqrt{E(X^2)-\mu^2}$

Properties

- $Var(k)=0$
- $Var(kX)=k^2Var(X)$
- $\text{Var}(X+k)=\text{Var}(X)$
- $\text{Var}(kX+b)=k^2\text{Var}(X)$
- $\sigma(kX)=\sigma_{kx}=|k|\sigma_x$

Example:

An r.v X has the following distribution:

X	1	2	3	4	5
P(X=x)	0.1	0.3	0.2	0.3	0.1

Find E(X), Var X and σ_X

Solution

a) $E(X)=\sum_{all\ x} xP(X=x)=(1\times 0.1)+(2\times 0.3)+(3\times 0.2)+(4\times 0.3)+(5\times 0.1)=3$

b) $Var\ X = E(X^2) - [E(X)]^2$

$$E(X^2) = \sum_{all\ x} x^2 P(X = x) = (1\times 0.1) + (4\times 0.3) + (9\times 0.2) + (16\times 0.3) + (25\times 0.1) = 10.4$$

So, that the $Var(X) = 10.4 - 3^2 = 1.4$ and $\sigma_x = \sqrt{1.4} = 1.1832$

Application activity 5.2.1

A financial adviser suggests that his client select one of two types of bonds in which to invest 5 000 000frw. Bond X pays a return of 4% and has a default rate of 2%. Bond Y has a $2\frac{1}{2}\%$ return and a default rate of 1%. Find the expected rate of return and decide which bond would be a better investment. When the bond defaults, the investor loses all the investment.

5.2.2 Binomial Distribution

Learning activity 5.2.2

With your potential explanations and examples to fit the following four conditions:

- A fixed number of trials
- Each trial is independent of the others
- There are only two outcomes
- The probability of each outcome remains constant from trial to trial.

Binomial distribution is calculated by multiplying the probability of success raised to the power of the number of successes and the probability of failure raised to the power of the difference between the number of successes and the number of trials.

Let consider an experiment with independent trials and only two possible outcomes. We call one of the successful outcome and the probability of its occurring is $P(s) = p$. The other outcome is called fail outcome and the probability of its occurring is $P(f) = q = 1 - p$. The probability for getting exactly k successes in n trials is $P_k = B(k, n, p) = B(n, p) = \binom{n}{k} p^k q^{n-k}$ *and* $k = 0, 1, 2, \ldots, n$

Or $P(k) = \frac{n!}{(n-k)!k!} p^k q^{n-k}$

Then the probability density function variable X is given by

$$f(x) = P(X = x) = \binom{n}{k} p^k q^{n-k} \text{ and } k = 0, 1, 2, \ldots, n$$

We write $X \sim Bin(n, p)$ and read "X is a random variable following the binomial distribution with parameter n and p".

Example:

1. A survey found that one out of five customers say he or she has bought items at Kigali Supermarket in any given month. If 10 customers are selected at random, find the probability that exactly 3 customers will have bought in Kigali supermarket last month.

Solution:

In this case, $n = 10, X = 3, p = \frac{1}{5}$ *and* $q = \frac{4}{5}$. Hence, $P(3) = \frac{10!}{(10-3)!3!}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7 = 0.201$

Mean, Variance, and Standard Deviation for binomial distribution

The mean, variance, and standard deviation of a binomial distribution are extremely easy to find. Mean: $\mu = np$. Variance: $\sigma^2 = npq$; Standard deviation: $\sigma = \sqrt{npq}$

2. The Statistical Bulletin published by Metropolitan Life Insurance Co. reported that 2% of all American births result in twins. If a random sample of 8000 births is taken, find the mean, variance, and standard deviation of the number of births that would result in twins.

Solution:

This is a binomial situation, since a birth can result in either twins or not twins (i.e., two outcomes). $\mu = np = (8000)(0.02) = 160$

$$\sigma^2 = npq = (8000)(0.02)(0.98) = 156.8$$

$$\sigma = \sqrt{npq} = \sqrt{156.8} = 12.5$$

For the sample, the average number of births that would result in twins is 160, the variance is 156.8, or 157, and the standard deviation is 12.5, or 13 if rounded.

Application activity 5.2.2

1. A multiple-choice test with 20 questions has five possible answers for each question. A completely unprepared student picks the answers for each question at random and independently. Suppose X is the number of questions that the student answers correctly. Calculate the probability that:

a) The student gets every answer wrong.
b) The student gets every answer right.
c) The student gets 8 right answers.

2. An examination consisting of 10 multiple choice questions is to be done by a candidate who has not revised for the exam. Each question has five possible answers out of which only one is correct. The candidate simply decides to guess the answers.

i. What is the probability that the candidate gets no answer correct?

ii. What is the probability that the student gets two correct answers?

3. Suppose you took three coins and tossed them in the air.

a) show all the possible outcomes when the coin is tossed
b) What is the probability that all three will come up heads?
c) What is the probability of 2 heads?
d) Show the probability distribution on a graph (use the number of heads)

5.2.3 Bernoulli distribution

Learning activity 5.2.3

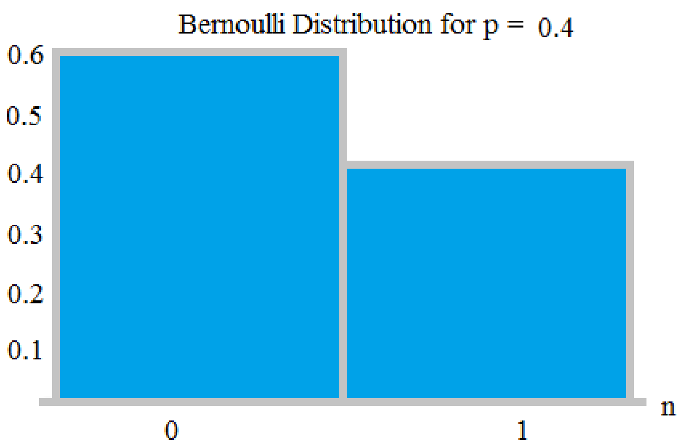
On your own, can you suggest the Bernoulli examples, using the binomial experiment shown above? What are your differences and Similarities?

A **Bernoulli distribution** is a discrete probability distribution for a Bernoulli trial. A **Bernoulli trial** is one of the simplest experiments you can conduct. It's an experiment where you can have one of two possible outcomes. For example, "Yes" and "No" or "Heads" and "Tails.

A random experiment that has only two outcomes (usually called a "Success" or a "Failure"). For example, the probability of getting a heads (a "success") while flipping a coin is 0.5. The probability of "failure" is 1 – P (1 minus the probability of success, which also equals 0.5 for a coin toss). It is a special case of the binomial distribution for $n = 1$. In other words, it is a binomial distribution with a single trial (e.g. a single coin toss).

For example

In the following Bernoulli distribution, the probability of success (1) is 0.4, and the probability of failure (0) is 0.6



The probability density function (PDF) for this distribution is $p^x(1-p)^{1-x}$, which can also be written as

$$p(n) = \begin{cases} 1-p, & for\ n = 0 \\ p & for\ n = 1 \end{cases}$$

The expected value for a random variable, X, for a Bernoulli distribution is

$\mathbf{E[X] = p}$, For example, if p = .04, then E[X] = 0.04.

The variance of a Bernoulli random variable is: $\mathbf{Var[X] = p(1 - p)}$.

Application activity 5.2.3

Provide any examples describing discrete not continuous data resulting from an experiment known as **Bernoulli process/trials** in your areas of interest as accounting professional.

5.2.4 Poison distribution

Learning activity 5.2.4

Describe the characteristics of the following examples:

a) The number of cars arriving at a service station in 1 hour;
b) The number of accidents in 1 day on a particular stretch of high way.
c) The number of visitors arriving at a party in 1 hour,
d) The number of repairs needed in 10 km of road,
e) The number of words typed in 10 minutes.
f) Why these skills and knowledge are needed in Accounting professional.

Considering the binomial distribution, the values of p and q and n are given. If there are cases where p is very small and n is very large, then calculation involved will be long. Such cases will arise in connection with rare events, for example.

- Persons killed in road accidents;
- The number of defective articles produced by a quality machine;
- The number of mistakes committed by a good typist, per page;
- The number of persons dying due to rare disease or snake bite;
- The number of accidental deaths by falling from trees or roofs etc.

In all these cases we know the number of times an event happened but not how many times it does not occur. Events of these types are further illustrated below:

1. It is possible to count the number of people who died accidently by falling from trees or roofs, but we do not know how many people did not die by these accidents.
2. It is possible to know or to count the number of earth quakes that occurred in an area during a particular period of time, but it is, more or less, impossible to tell as to how many times the earth quakes did not occur.
3. It is possible to count the number of goals scored in a foot-ball match but cannot know the number of goals that could have been but not scored.
4. It is possible to count the lightning flash by a thunderstorm but it is impossible to count as to how many times, the lightning did not flash etc.

Thus *n*, the total of trials in regard to a given event is not known, the binomial distribution is inapplicable,

Poisson distribution is made use of in such cases where *p* is very small. We mean that the chance of occurrence of that event is very small. The occurrence of such events is not haphazard. Their behavior can also be explained by mathematical law. Poisson distribution may be obtained as a limiting case of binomial distribution.

Definition

A random variable X taking values $0, 1, 2, 3, 4, \ldots, +\infty$ has a ***Poisson distribution***

$$p(X-x)=\frac{e^{-\lambda}\lambda^{x}}{x!} \text{ for } x = 0, 1, 2, \ldots .$$

Where the letter e is a constant approximately equal to 2.7183.

We write **X~ Poisson (λ).**

As required for a probability function, it can be shown that the probabilities

$P(X = x)$ all sum to 1.

Examples:

1. While checking the galley proofs of the first four chapters of our last book, the authors found 1.6 printer's errors per page on average. We will assume the errors were occurring randomly according to a Poisson process. Let X be the number of errors on a single page. Then

$X \sim$ Poisson ($\lambda = 1.6$). We will use this information to calculate a large number of probabilities.

a) The probability of finding no errors on any particular page is P(X = 0) = $e^{-1.6}$=0.2019.

b) The probability of finding 2 errors on any particular page is $P(X=2)=\frac{e^{-1.6}(1.6)^2}{2!}=0.2584$

c) The probability of no more than 2 errors on

a page is $P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = 0.7833$

d) The probability of more than 4 errors on a page is

$P(X > 4) = P(X=5) + P(X=6) + P(X=7) + P(X=8) + \ldots$

so if we tried to calculate it in a straightforward fashion, there would be an infinite number of terms to add. However, if we use $P(A) = 1 - P(\bar{A})$ we get $P(X > 4) = 1 - P(X \le 4) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)]$

$= 1 - (0.2019 + 0.3230 + 0.2584 + 0.1378 + 0.0551)$

$= 1 - 0.9762 = 0.0238$

e) Let us now calculate the probability of getting a total of 5 errors on 3 consecutive pages.

f) Let Y be the number of errors in 3 pages. The only thing that has changed is that we are now looking for errors in bigger units of the manuscript so that the average number of events per unit we should use changes from1.6 errors per page to 3 x 1.6 = 4.8 errors per 3 pages.

Thus, Y ~ Poisson (λ = 4.8). $P(Y=5)=\frac{e^{-4.8}(4.8)^5}{5!}=0.1747$

2. If there are 500 customers per eight-hour day in a check-out lane, what is the probability that there will be exactly 3 in line during any five-minute period?

Solution:

The expected value during any one five minute period would be $\frac{500}{96}$ = 5.2083333. The 96 is because there are 96 five-minute periods in eight hours. So, you expect about 5.2 customers in 5 minutes and want to know the probability of getting exactly 3. $P\left(3;\frac{500}{96}\right)=\frac{e^{\left(\frac{-500}{96}\right)}\left(\frac{500}{96}\right)^3}{3!}=0.1288$ (approx)

The mean of the Poisson distribution is: ***E*(*X*) = λ.**

The variance of the Poisson distribution is: ***Var* (*X*) =** λ.

Note that the variance is equal to the mean.

3. Guests coming to attend the graduation ceremony at the university arrive at the average rate of 2.5 per minute. Determine the probability that 5 guests will arrive in a one minute period.

Answer: The Poisson probability function is given by

$$f(x)=\frac{\mu^x e^{-\mu}}{x!}, \quad for \quad \mu=2.5, \quad x=5, \quad Therefore \quad f(5)=\frac{(2.5)^5 e^{-2.5}}{5!}=0.0668$$

Application activity 5.2.4

1. A lecturer of statistics has observed that the number of typing errors in new edition of text books varies considerably from book to book. After some analysis we conclude that the number of errors is a Poisson distribution with a mean 1.5 per 100 pages. The lecturer randomly selects 100 pages of a new book. What is the probability that there are no typing errors?
2. If there are 200 typographical errors randomly distributed in a 500-page manuscript, find the probability that a given page contains exactly 3 errors.
3. The mean number of bacteria per millimeter of a liquid is known to be 4. Assuming that the number of bacteria follows a Poisson distribution, find the probability that 1 ml of liquid there will be: No bacteria, 4 bacteria, less than 3 bacteria

5.2.5 Continuous Probability Distributions

Learning activity 5.2.5

Refer to the integration properties to answer the following questions:

1. What are your observations on the following function? Where

$$f(x)=\begin{cases}\frac{3}{4}\left(1+x^2\right), & 0\le x\le 1\\ 0, & otherwise\end{cases}$$

2. Find Expectation, variance and standard deviation.

A continuous random variable is a theoretical representation of a continuous variable such as height, mass or time.

Examples

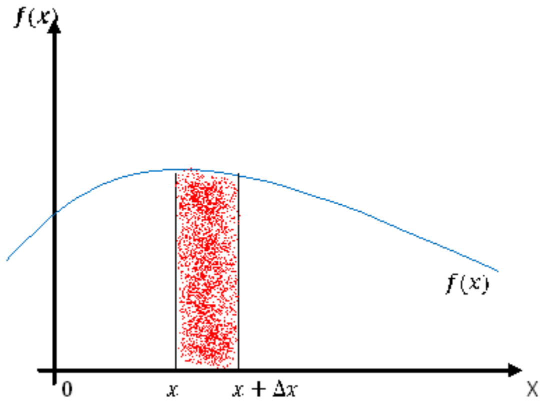
- the amount of time to complete a task.
- Heights of trees in a school garden
- Daily temperature
- Interest earned daily on bank loans per person

If X is a continuous random variable, then

$$p(a<X\le b)=p(a\le X\le b)=p(a\le X<b)=p(a<X<b)=\int_a^b f_X(x)dx=F_X(b)-F_X(a)$$

Probability Density Function (PDF) for continuous random variables.

The probability mass function of the continuous random variable X is called **probability density function (PDF).** It is denoted by ***f(*** x ***)*** or ***p(*** x ***)*** where ***f(x*** is the probability that the random variable X takes the value between x and $x+\Delta x$ where Δx is a very small change in X**.**



If X is a continuous random variable, then the probability that the value of X will fall between the values a and b is given by the area of the region lying below the graph of ***f(x)*** and above the x-axis between a and b. i.e. . $p(a \le X \le b) + \int_a^b f(x)dx$

Where a and b are the points between +∞ and -∞ , the quantity $f(x)dx$ is called **Probability differential.**

Note: For any probability density function:

- $f(x) \ge 0$ for all x
- The total area under the graph of $f(x)$ must be 1.
- $P(X = c) = \int_c^c f(x)dx = 0$ Where c is any constant.

Examples:

1. A continuous random variable X which can assume between 2 and 8 inclusive has a probability density function $f(x) = c(x+3)$ where c is a constant.

Find:

i. The value of c

ii. $P(3 < X < 5)$

iii. $P(X \ge 4)$

Solution:

(i) $\int_2^8 c(x+3)dx = 1 \Leftrightarrow c = \frac{1}{48}$ so that $f(x) = \frac{1}{48}(x+3), 2 \le x \le 8$

(ii) $P(3 < X < 5) = \int_3^5 \frac{1}{48}(x+3)dx = \frac{7}{24}$

(iii) $P(X \geq 4) = \int_4^8 \frac{1}{48}(x+3)\,dx = \frac{3}{4}$

If f(x) is the probability density function on $(-\infty, +\infty)$, then:

$$1. \int_{-\infty}^{+\infty} f(x)dx = 1$$

$$2. E(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

$$3. Var(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx - E^2(X)$$

$$4. \sigma_x = \sqrt{Var(X)}$$

Example

The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{3}{4}\left(1+x^2\right), & 0 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

Find the expectation, the variance and the standard deviation of X.

Solution

$$E(X) = \int_{allx} f(X)dx = \frac{3}{4}\int_0^1 x(1+x^2)dx = \frac{3}{4}\int_0^1 (x+x^3)dx = \frac{3}{4}\left[\frac{x^2}{2}+\frac{x^4}{2}\right]_0^1 = \frac{3}{4}\left(\frac{1}{2}+\frac{1}{4}\right)$$

$$= \frac{3}{4}\left(\frac{3}{4}\right) = \frac{9}{16} = 0.5625$$

But $E(X^2) = \int_{all x} x^2 f(x)\,dx = \frac{3}{4}\int_0^1 (x^2 + x^4)dx = \frac{3}{4}\left[\frac{x^3}{3} + \frac{x^5}{5}\right] = \frac{3}{4} \times \frac{8}{15} = \frac{2}{5} = 0.4$

$$Var(X) = \int_{all\ x} x^2 f(x)dx - \mu^2$$

Then, $Var\ X = 0.4 - (0.5625)^2 = 0.0835$

$$\sigma_x = \sqrt{Var\ X} = 0.289$$

Application activity 5.2.5

Find the constant c such that the function $f(x) = \begin{cases} cx^2, & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$ is a density function.

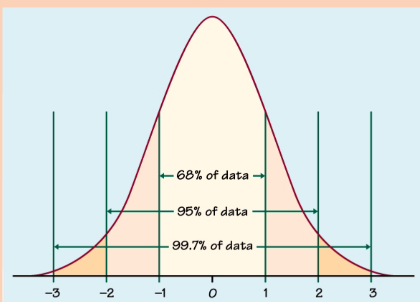
a) Compute $P(1 < X < 2)$

b) How can you compute $P(-\infty < X < +\infty)$?

c) Compare results in (a) and (b) statistically.

5.2.6 Normal distribution

Learning activity 5.2.6



Explain clearly the results shown above, how these findings are linked to probability distributions? Can you compare descriptive statistics and this probability distribution as an accountant? How?

Definitions

A **normal distribution** is a continuous, symmetric, bell-shaped distribution of a variable.

Standard Error or the Mean: The standard deviation of the sampling distribution of the sample means. It is equal to the standard deviation of the population divided by the square root of the sample size.

Standard Normal Distribution: A normal distribution in which the mean is 0 and the standard deviation is 1. It is denoted by z and Z score is used to represent the standard normal distribution.

The formula for the standard normal distribution is

$$f(x) = \frac{e^{\left(\frac{-z^2}{2}\right)}}{\sqrt{2\pi}}.$$

All normally distributed variables can be transformed into the standard normally distributed variable by using the formula for the standard score:

$$z = \frac{Value - mean}{Standard\ deviation} \text{ or } z = \frac{X - \mu}{\sigma}$$

- A **standard normal distribution table** shows a cumulative probability associated with a particular z-score. Table rows show the whole number and tenths place of the z-score. Table columns show the hundredths place. The cumulative probability (often from minus infinity to the z-score) appears in the cell of the table.
- For example, a section of the standard normal table is reproduced below. To find the cumulative probability of a z-score equal to -1.31, cross-reference the row of the table containing -1.3 with the column containing 0.01. The table shows that the probability that a standard normal random variable will be less than -1.31 is 0.0951; that is, P(Z < -1.31) = 0.0951.

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
...	...	...	...	...	...	...	...	...	...	...
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0722	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
...	...	...	...	...	...	...	...	...	...	...
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

These probabilities are easy to compute from a normal distribution. Their full table attached in appendix. For examples:

- Find P(Z > a). The probability that a standard normal random variable (z) is greater than a given value (a) is easy to find. The table shows the P(Z < a).
- The P(Z > a) = 1 - P(Z < a). Suppose, for example, that we want to know the probability that a z-score will be greater than 3.00. From the table (see above), we find that P(Z < 3.00) = 0.9987. Therefore, P(Z > 3.00) = 1 - P(Z < 3.00) = 1 - 0.9987 = 0.0013.
- Find P(a < Z < b). The probability that a standard normal random variables lies between two values is also easy to find. The P(a < Z < b) = P(Z < b) - P(Z < a).

Example, suppose we want to know the probability that a z-score will be greater than -1.40 and less than -1.20. From the table (see above), we find that P(Z < -1.20) = 0.1151; and P(Z < -1.40) = 0.0808. Therefore, P(-1.40 < Z < -1.20) = P(Z < -1.20) - P(Z < -1.40) = 0.1151 - 0.0808 = 0.0343.

The **mean and variance** of the normal random variable X are:

$$E(X) = \mu_x = \mu,\ Var(X) = \sigma_x^2 = \sigma^2$$

In general we have: $X \sim N\left(\mu, \sigma^2\right)$ Such that the density distribution is given by this mathematical expression: $f(X) = \frac{1}{\sqrt{2\pi}} e\left\{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$

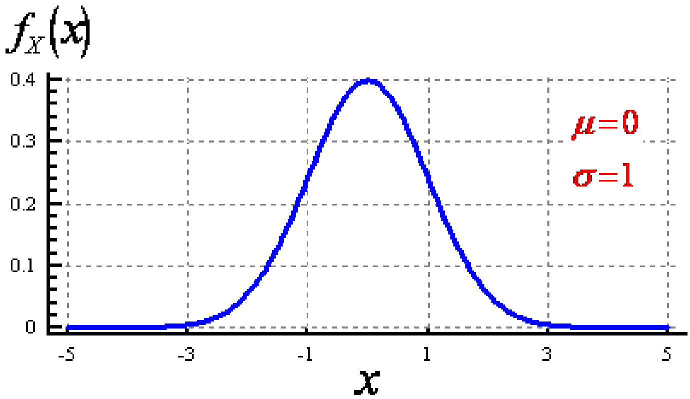
Properties

$$1.\, p[a < X < b] = p[X < b] - p[X < a]$$

$$2.\, p[X > a] = 1 - p[X < a]$$

$$3.\, p[X < -a] = 1 - p[X < a]$$

$$4.\, p\left(a < X < b / X > c\right) = \frac{p[(a < X < b)] \cap (X > c)}{p[X > c]}$$



Examples:

1. The age of the subscribers to a newspaper has a normal distribution with mean 50 years and standard deviation 5 years. Compute the percentage of subscribers who are less than 40 years old and the percentage of who are between 40 and 60 years old.

Answer

Let X denote the age of a subscriber. We have $X \sim N\left(\mu, \sigma^2\right)$, $\mu = 50$, $\sigma = 5$.

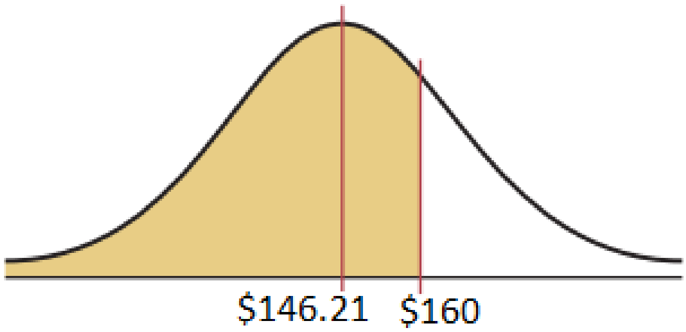
Therefore, $P(X < 40) = P\left(\frac{X-50}{5} < \frac{40-50}{5}\right) = P(Z < -2) = 1 - \Phi(2) = 0.02275 = 2.27\%$

$$P(40 \le X \le 60) = P\left(\frac{40-50}{5} \le \frac{X-50}{5} \le \frac{60-50}{5}\right) = P(-2 \le Z \le 2) = 2\Phi(2) = 0.9545 = 95.4\%$$

2. A survey by the National Statistics of Rwanda found that women spend on average \$146.21 for the Christmas holidays. Assume the standard deviation is \$29.44. Find the percentage of women who spend less than \$160.00. Assume the variable is normally distributed.

Solution:

Step 1 Draw the figure and represent the area as shown in the following figure

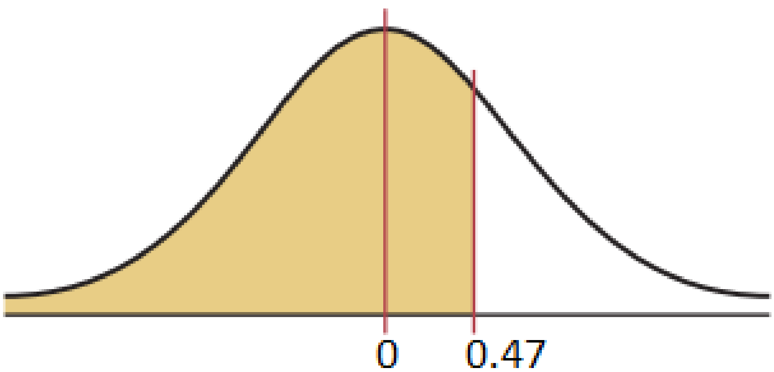


Step 2

Find the z value corresponding to $160.00.

$$z = \frac{X - \mu}{\sigma} = \frac{\$160 - \$146.21}{\$29.44} = 0.47$$

Hence $160.00 is 0.47 of a standard deviation above the mean of $146.21, as shown in the z distribution in the following figure



Step 3

Find the area, using the table which is on ***appendix*** of this student book. The area under the curve to the left of z = 0.47 is 0.6808. Therefore 0.6808, or 68.08%, of the women spend less than $160.00 at Christmas time.

Application activity 5.2.6

1. Let $Z \sim N(0,1)$. Find the values of $P(-1 < Z < 3)$ and $P(-3 < Z < 3)$.

2. One Agency of communication in Rwanda reports that the average time it takes to respond to an emergency call is 25 minutes. Assume the variable is approximately normally distributed and the standard deviation is 4.5 minutes. If 80 calls are randomly selected, approximately how many will be responded to in less than 15 minutes?

5.3. Applications of probability and probability distributions in finance, economics, businesses, and production related domain.

Learning activity 5.3

1. The Trucks packed goods stayed at MAGERWA for under clearing process in certain days. The number of days shown in the distribution are displayed in the table below:

Number of days stayed	Frequency
3	15
4	32
5	56
6	19
7	5
Total	127

Find these probabilities: (a) A Truck stayed exactly 5 days. (b) A Truck stayed less than 6 days. (c) A Truck stayed at most 4 days. (d) A Truck stayed at least 5days.

Probability is essential in all aspects of human activities. Through this, we can calculate families welfare, get information on different aspects of life and planification in order to predict the future. It is very important to incorporate probability with their properties looking at distributions to adresss these real-world issues. Therefore, probability play a great role for decision makers.

Examples:

1. In Economics, A welfare of families depending on family planning. A woman planning her family considers the following schemes on the assumption that boys and girls are equally likely at each delivery:

   a) Have three children.

   b) Bear children until the first girl is born or until three are born, whichever is sooner, and then stop.

   c) Bear children until there is one of each sex or until there are three, whichever is sooner, and then stop.

Let B_i denote the event that i boys are born, and let C denote the event that more girls are born than boys. Find $P(B_1)$ and P(C) in each of the cases (a) and (b).

Solution:

a) If we do not consider order, there are four distinct possible families: BBB, GGG, GGB, and BBG, but these are not equally likely. With order included, there are eight possible families in this larger sample space:

(1) {BBB; BBG; BGB; GBB; GGB; GBG; BGG; GGG}= Ω

and by symmetry they are equally likely. Now, $P(B_1)=\frac{3}{8}$ and $P(c)=\frac{1}{2}$. The fact that $P(c)=\frac{1}{2}$ is also clear by symmetry.

Now consider (b). There are four possible families: F1 = G, F2 = BG, F3 = BBG, and F4 = BBB. Once again, these outcomes are not equally likely, but as we have now done several times we can use a different sample space. One way is to use the sample space in (1), remembering that if we do this then some of the later births are pretended. The advantage is that outcomes are equally likely by symmetry. With this choice, F2 corresponds to $\{BGG \cup BGB\}$ and $P(B_1)=P(F_2)=\frac{1}{4}$.

Likewise, $F_1=\{GGG \cup GGB \cup GBG \cup GBB\}$ and so $P(C)=\frac{1}{2}$.

2. In any given year, the probability that a given male driver has a mishap entailing a claim from his insurance company is μ, independently of other years. The equivalent probability in female drivers is λ. Assume

there are equal numbers of male and female drivers insured with the Assurance Association, which selects one of them at random.

a) What is the probability that the selected driver makes a claim this year?

b) What is the probability that the selected driver makes a claim in two consecutive years?

c) If the insurance company picks a claimant at random, what is the probability that this claimant makes another claim in the following year?

Solution:

a) Let A_1 and A_2 be the events that a randomly chosen driver makes a claim in each of the first and second years. Then conditioning on the sex of the driver (M or F) gives $P(A_1) = P(A_1 | M)P(M) + P(A_1 | F)P(F) = \frac{1}{2}(\mu + \lambda)$ because $P(F) = P(M) = \frac{1}{2}$.

b) Likewise,

$$P(A_1 \cap A_2) = P(A_1 \cap A_2 | M)P(M) + P(A_1 \cap A_2 | F)P(F) = \frac{1}{2}(\mu^2 + \lambda^2).$$

c) By definition, $P(A_2 | A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \mu^2 + \lambda^2$.

3. The fixed-price dinner at One Restaurant provides the following choices:

Appetizer: soup or salad

Entree: baked chicken, broiled beef patty, baby beef liver, or roast beef au jus

Dessert: ice cream or cheesecake.

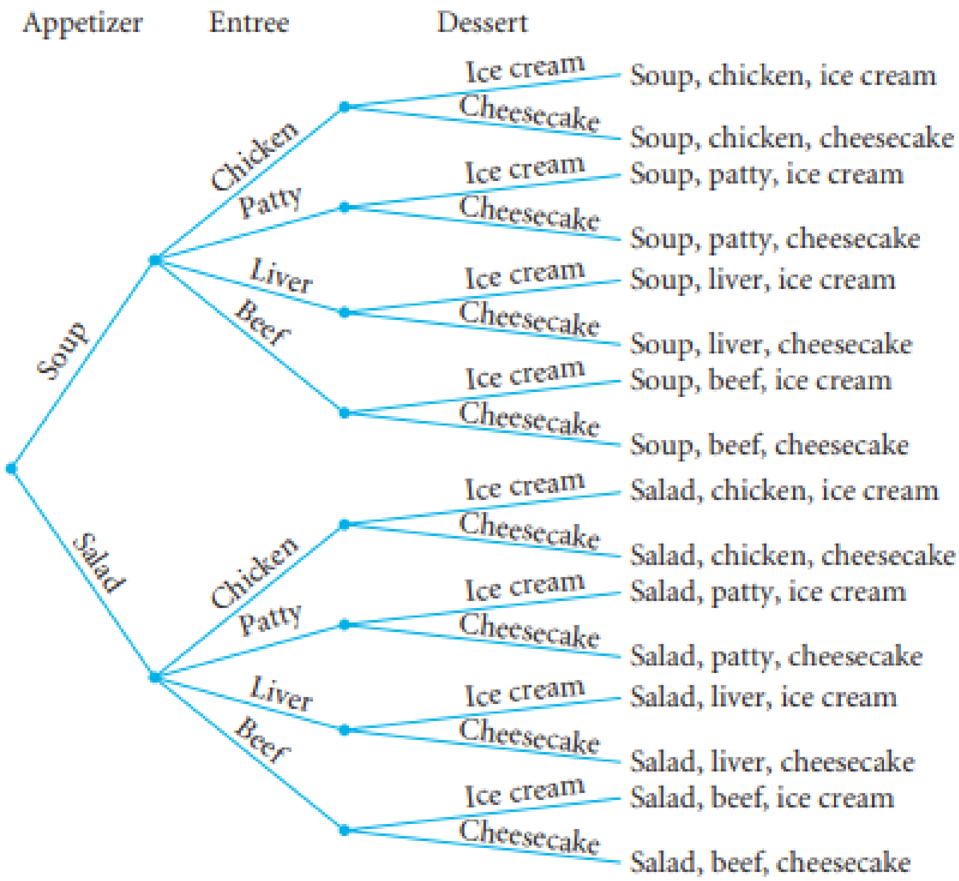
How many different meals can be ordered?

Solution:

Ordering such a meal requires three separate decisions:

Choose an Appetizer: 2 choices. **Choose an Entrée:** 4 choices. **Choose a Dessert**: 2 choices.

Look at the tree diagram in the following figure. We see that, for each choice of appetizer, there are 4 choices of entrees. And for each of these $2 \times 4 = 8$ choices, there are 2 choices for dessert. A total of $2 \times 4 \times 2 = 16$ different meals can be ordered.


Appetizer
Entree
Dessert
Soup
Salad
Chicken
Patty
Liver
Beef
Ice cream
Cheesecake
Soup, chicken, ice cream
Soup, chicken, cheesecake
Soup, patty, ice cream
Soup, patty, cheesecake
Soup, liver, ice cream
Soup, liver, cheesecake
Soup, beef, ice cream
Soup, beef, cheesecake
Salad, chicken, ice cream
Salad, chicken, cheesecake
Salad, patty, ice cream
Salad, patty, cheesecake
Salad, liver, ice cream
Salad, liver, cheesecake
Salad, beef, ice cream
Salad, beef, cheesecake

Application activity 5.3

1. An automatic Shaw machine fills plastic bags with a mixture of beans, broccoli, and other vegetables. Most of the bags contain the correct weight, but because of the slight variation in the size of the beans and other vegetables, a package might be slightly underweight or overweight. A check of 4000 packages filled in the past month revealed:

Weight	Event	Number of packages	Probability of occurrence
Underweight	A	100	100/4000=0.025
Satisfactory	B	3600	3600/4000=0.9
Overweight	C	300	300/4000=0.75

What is the probability that a particular package will be either underweight or overweight?

2. A baker must decide how many specialty cakes to bake each morning. From past experience, she/he knows that the daily ***demand for cakes ranges from 0 to 3.Each cake costs $3.00*** to produce and ***sells for $8.00,*** and any unsold cakes are thrown in the garbage at the end of the day.

   a) Set up a payoff table to help the baker decide how many cakes to bake

   b) Assuming probability of each event is equal, determine the expected value of perfect information.

5.4. End unit assessment

1. Which of the following experiments does not have equally likely outcomes?

   a) Choose a number at random between 1 and 7,

   b) Toss a coin,

   c) Choose a letter at random from the word SCHOOL,

   d) None of the above.

2. Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?

3. In a class, there are 15 boys and 10 girls. Three students are selected at random. What is the probability that 1 girl and 2 boys are selected?

4. One card is drawn at random from a pack of 52 cards. What is the probability that the card drawn is a face card (Jack, Queen and King only)?

5. The letters of the word FACETIOUS are arranged in a row. Find the probability that

   a) the first 2 letters are consonants,

   b) all the vowels are together.

6. At a certain school, the probability that a student takes Auditing and Taxation is 0.087. The probability that a student takes Auditing is 0.68. What is the probability that a student takes Taxation given that the student is taking Auditing?

7. A car dealership is giving away a trip to Akagera National Park to one of their 120 best customers. In this group, 65 are women, 80 are married and 45 married women. If the winner is married, what is the probability that it is a woman?

8. For married couples living in a certain suburb the probability that the husband will vote on a bond referendum is 0.21, the probability that his wife will vote in the referendum is 0.28, and the probability that both the husband and wife will vote is 0.15. What is the probability that

a) at least one member of a married couple will vote ?

b) a wife will vote, given that her husband will vote ?

c) a husband will vote, given that his wife does not vote ?

9. The probability that a patient recovers from a delicate heart operation is 0.8. What is the probability that

a) exactly 2 of the next 3 patients who have this operation will survive ?

b) all of the next 3 patients who have this operation survive ?

10. In a certain college, 5% of the men and 1% of the women are taller than 180 *cm*. Also, 60% of the students are women. If a student is selected at random and found to be taller than 180 *cm*, what is the probability that this student is a woman?

11. Suppose that entire output of a factory is produced on three machines. Let B_1 denote the event that a randomly chosen item was made by machine 1, B_2 denote the event that a randomly chosen item was made by machine 2 and B_3 denote the event that a randomly chosen item was made by machine 3. Let *A* denote the event that a randomly chosen item is defective.

i. Use conditional probability formula and give the relation should be used to find the probability that the chosen item is defective, $P(A)$, given that it is made by machine 1 or machine 2 or machine 3.

ii. If we need the probability that the chosen item is produced by machine 1 given that is found to be defective, i.e $P(B_1 | A)$, give the formula for this conditional probability. Recall that $P(B_i \cap A)$ can be written as $P(A | B_i)P(B_i)$. Do the same if the item is produced by machine 2 and by machine 3. Give the general formula if the item is produced by machine *i* (*i* from 1 to 3)

Bibliography

1. *Allan G. B. (2007). Elementary statistics: a step by step approach, seventh edition, Von Hoffmann Press, New York.*

2. *David F. Groebner, (2008),Business Statistics-Decision Making, 7th edition. Prentice Hall.*

3. *David R. (2000). Higher GCSE Mathematics, revision and Practice. Oxford University Press, UK.*

4. *David Rayner, Higher GCSE Mathematics, Oxford University Press 2000*

5. *Douglas A. Lind, William G. Marchal and Robert D. Mason, (2002), Statistical Techniques in Business & Economics,11th edition, New York, McGraw-Hill Irwin.*

6. *Frank Ebos, Dennis Hamaguchi, Barbana Morrison & John Klassen, Mathematics Principles & Process, Nelson Canada A Division of International Thomson Limited 1990*

7. *Geoff Mannall & Michael Kenwood, Pure Mathematics 2, Heinemann Educational Publishers 1995*

8. *George B. Thomas, Maurice D. Weir & Joel R. Hass, Thomas' Calculus Twelfth Edition, Pearson Education, Inc. 2010*

9. *Gerald Keller, Brian Warrack, (2003), Statistics for Management and Economics, 6th edition, Duxbury Thomson Learning, Pacific Grove.*

10. *https://www.brainkart.com. Business mathematics 11th std. cost function from marginal cost function*

11. *https://www.google.com/search? q=application +of+probability+ in+business&oq=&aqs=chrome. 0.35i39i362l5 j46i39i362j35i39i3 62l2.257365776 j0j15&sourceid = chrome&ie=UTF-8, retrieved 4th, Feb, 2024*

12. *https://www.johndcook.com/blog/forecasting/ , retrieved 4th, Feb 2024*

13. *J CRAWSHAW, J CHAMBERS: A concise course in A-Level statistics with worked examples, Stanley Thornes (Publishers) LTD, 1984.*

14. *J. CRAWSHAW and J. CHAMBERS 2001. A concise course in Advanced Level Statistics with worked examples 4th Edition. Nelson Thornes Ltd, UK.*

15. *J. Sadler, D. W. S. Thorning: Understanding Pure Mathematics, Oxford University Press 1987.*

16. *James T. McClave, P. George Benson and Terry Sincich, (2008), Statistics for Business and Economics, Prentice Hall.*

17. *James T. McClave, P.George Benson. Statistics for Business and Economics. USA, Dellen Publishing Company, a division of Macmillan, Inc 1988.*

18. *James T. McClave, P.George Benson. Statistics for Business and Economics. USA, Dellen Publishing Company, a division of Macmillan, Inc 1988.*

19. *Jean-Philippe RÉAU and Gérard CHAUVAT, (1999), Probabilités & Statistiques. Armand Colin.*

20. *John bird. (2005). Basic engineering mathematics. 4th Edition. Linacre House, Jordan Hill, Oxford OX2 8DP*

21. *K.A. Stroud. (2001). Engineering mathematics. 5th Edition. Industrial Press, Inc, New York*

22. *Ngezahayo E.(2016). Advanced Mathematics for Rwanda secondary Schools, Learners' book 4, Fountain publishers, Kigali.*

23. *Ngezahayo E.(2016). Advanced Mathematics for Rwanda secondary Schools, Learners' book 4, Fountain publishers, Kigali.*

24. *Ngezahayo E.(2016). Subsidiary Mathematics for Rwanda secondary Schools, Learners' book 4, Fountain publishers, Kigali.*

25. *Ngezahayo E.(2017). Advanced Mathematics for Rwanda secondary Schools, Learners' book 5, Fountain publishers, Kigali.*

26. *Ngezahayo E.(2017). Subsidiary Mathematics for Rwanda secondary Schools, Learners' book 5, Fountain publishers, Kigali.*

27. *Ngezahayo E.(2017). Subsidiary Mathematics for Rwanda secondary Schools, Learners' book 5, Fountain publishers, Kigali.*

28. *Ngezahayo E.(2018). Advanced Mathematics for Rwanda secondary Schools, Learners' book 6, Fountain publishers, Kigali.*

29. *Ngezahayo E.(2018). Advanced Mathematics for Rwanda secondary Schools, Learners' book 6, Fountain publishers, Kigali.*

30. *Norman L. Johnson, Samuel Kotz and Adrienne W. Kemp, (1992). Univariate Discrete Distributions, 2nd Edition. John Wiley & Son.*

31. *Peter S. (2005). Mathematics HL&SL with HL options, Revised edition. Mathematics Publishing PTY. Limited.*

32. *Peter Smythe: Mathematics HL & SL with HL options, Revised Edition, Mathematics Publishing Pty. Limited, 2005.*

33. *REB (2020), Mathematics for TTCs year three, Science and Mathematics Education (SME), Kigali, Rwanda*

34. *REB (2020), Mathematics for TTCs year three, social studies education (SSE), Kigali, Rwanda*

35. *REB. (2015). Subsidiary Mathematics Syllabus, MINEDUC, Kigali, Rwanda.*

36. *REB. (2019). Mathematics Syllabus for TTC-Option of LE, MINEDUC, Kigali Rwanda.*

37. *S. Brown and W. K. Wong, (2015). Probability and Statistics with Applications in Finance and Economics, Hindawi, The Scientific World Journal 2015:618785, DOI: 10.1155/2015/618785, License CC BY*

38. *Sadler A. J & Thorning D.W. (1997). Understanding Pure mathematics, Oxford university press, UK.*

39. *Seymour Lipschutz, PhD. & Marc Lipson, PhD. (2007). Discrete mathematics. 3rd edition.*

40. *Seymour Lipschutz. Schaum's outline of Theory and Problems of Finite Mathematics. New York, Schaum Publisher, 1966*

41. *Shampiyona Aimable : Mathématiques 6. Kigali, Juin 2005.*

42. *Wellars Banzi, and Nteziyaremye Stanislas, (2023). Business Research Methodology and Operations Research Lectures note, Republic of Rwanda, Kigali Independent University.*

APPENDIX

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997